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Vladimir L. Kharitonov

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Lyapunov Functionals and Matrices

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A nuestros amigos de Mexico con cariño i respeto

Preface

Although stability is one of the most studied topics in the theory of time-delay systems, the corresponding chapters of classic works on time-delay systems (see, e.g., [3, 23, 44]) do not include a comprehensive study of a counterpart of the classic Lyapunov theory for linear delay-free systems. The principal aim of this volume is to fill this gap and provide the reader with a detailed treatment of the basic concepts of the Lyapunov–Krasovskii approach to the stability analysis of linear time-delay systems.

There are two types of stability results. Results of the first type are obtained in the following manner. First, a positive-definite functional is selected, then its time derivative along the solutions of a system is computed, and, finally, some negativity conditions for the derivative are proposed. This is how the majority of LMI type stability conditions have been obtained. A good account of such stability results can be found in [16, 17, 31, 54, 58, 64]. The scheme to obtain results of the second type is different. First, a desired time derivative is selected, and then a functional with this time derivative along the solutions is computed. Finally, one needs to check whether or not the functional is positive definite. Usually, functionals obtained in this way are more complex than those used to derive results of the first type. But since these functionals are adjusted to the system under consideration, they provide more complete information about system behavior. It would be naive to expect that the second scheme could be successfully applied to general classes of time-delay systems. Of course, such results should be available for the case of linear time-delay systems.

This book is divided into two parts. The first part, consisting of four chapters, considers the case of retarded type time-delay systems. The first chapter of this part is of the compilation character. The chapter discusses such basic notions as initial conditions and system state. In the exposition of the existence and uniqueness results presented in this chapter we follow [19]. Classical stability results based on the Lyapunov–Krasovskii approach are presented in a form inspired by [72].

In Chap. 2 the class of linear systems with one delay is studied. We start with a computation of the solutions of such systems. Then we explain in detail a general scheme used for the computation of Lyapunov functionals with a prescribed time derivative. Here matrix-valued functions that define these functionals are introduced. They are a counterpart of the classic Lyapunov matrices that appear as solutions of the classical Lyapunov matrix equation in the context of Lyapunov quadratic forms for the case of linear delay-free systems. We call them Lyapunov matrices for time-delay systems. A substantial part of this chapter is devoted to an analysis of the basic properties of Lyapunov matrices. Such issues as existence, uniqueness, and computation are treated. Next, we introduce Lyapunov functionals that admit quadratic lower and upper bounds. These are functionals of the complete type. Complete type functionals are then used to derive exponential estimates for the solutions of time-delay systems and robustness bounds for perturbed systems. The chapter ends with a brief historical survey, where the results of the principal contributors to the subject are presented.

The material presented in Chaps. 1 and 2 is recommended for an introductory course on the stability of time-delay systems. Such courses have been given for several years in the Department of Automatic Control at CINVESTAV in Mexico City and now in the Faculty of Applied Mathematics and Control Processes of Saint Petersburg State University in Russia.

In Chap. 3 we address the case of retarded type linear time-delay systems with multiple delays. Applying the scheme presented in the previous chapter, we obtain a general form of quadratic functionals with a prescribed time derivative along the solutions of such time-delay systems. A special system of matrix equations that defines the Lyapunov matrices is derived. It is shown that the special system admits a unique solution if and only if the spectrum of the time-delay system does not contain points arranged symmetrically with respect to the origin of the complex plane. This spectrum property is known as the Lyapunov condition. Two numerical schemes for the computation of Lyapunov matrices are presented. The first one is applicable to the case where time delays are multiple to a basic one. The other one allows one to compute approximate Lyapunov matrices in the case of general time delays. A measure that makes it possible to estimate the quality of an approximation is provided as well. Quadratic functionals of the complete type are defined, and several important applications of the functionals are presented in the final part of the chapter.

In Chap. 4 a linear retarded type system with distributed delay is studied. First, we introduce quadratic functionals and Lyapunov matrices for the system. Then we present the existence and uniqueness conditions for the matrices and provide some numerical schemes for the computation of the Lyapunov matrices. Finally, we derive a class of time-delay system with distributed delay for which Lyapunov matrices are solutions of a boundary value problem for an auxiliary system of linear delay-free matrix differential equations.

The second part of the book, comprising three chapters, is devoted to the case of neutral type time-delay systems. In Chap. 5 we extend the results presented in Chap. 1 to the case of neutral type time-delay systems. Issues of existence, uniqueness, and continuation of solutions of the initial value problem for such systems are discussed. Stability concepts and basic stability results obtained using the Lyapunov–Krasovskii approach, mainly in the form of necessary and sufficient conditions, are presented.

In Chap. 6 we consider the class of neutral type linear systems with one delay. We define the fundamental matrix of such a system and present the Cauchy formula for the solution of an initial value problem. This formula is used to compute a quadratic functional with a given time derivative along the solutions of the time-delay system. It is demonstrated that this functional is defined by a Lyapunov matrix for the time-delay system. A thorough analysis of the basic properties of this Lyapunov matrix is conducted. Complete type functionals are introduced, and various applications of the functionals are discussed.

The last chapter is dedicated to the case of neutral type linear systems with distributed delay. The structure of quadratic functionals with prescribed time derivatives along the solutions of such a system is defined, and the corresponding Lyapunov matrices are introduced. A system of matrix equations that defines the Lyapunov matrices is given. It is proven that under some conditions this system admits a unique solution. A class of systems with distributed delay for which Lyapunov matrices are the solutions of standard boundary value problems for an auxiliary system of linear matrix ordinary differential equations is presented. Complete type functionals are defined. It is shown that these functionals can be presented in a special form that is more convenient for the computation of lower and upper bounds for the functionals.

The book's bibliography does not pretend to cover all aspects of the stability analysis of time-delay systems. It includes entries that are closely related to the problems discussed in the book. More complete lists of literature can be found in [18, 23, 41, 43, 58].

To conclude this preface, I would like to acknowledge the fruitful collaboration and friendly support of my colleagues Alexei Zhabko, Diederich Hinrichsen, Sabine Mondie, Alexander Alexandrov, and Silviu-Iulian Niculescu. I greatly appreciate their comments and suggestions.

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Peterhof, Saint Petersburg, Russia Vladimir L. Kharitonov

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Part I Retarded Type Systems

Notations and Symbols

R	Field of real numbers
R^n	Space of n-vectors with entries in R
i	Imaginary unit, $i^2 = -1$
C	Field of complex numbers
C^n	Space of n-vectors with entries in C
$0_{n\times n}$	Zero $n \times n$ matrix
I	Identity matrix
$Re(s)$, $Im(s)$	Real and imaginary parts of a complex number $s \in C$
$\|x\|$	Euclidean (Hermitian) norm of a vector $x \in R^n$ $(x \in C^n)$
$\|A\|$	Induced norm of a matrix A, $\|A\| = \max_{\|x\|=1} \|Ax\|$
$C([-h,0],R^n)$	Space of R^n-valued continuous functions on $[-h,0]$
$PC([-h,0],R^n)$	Space of R^n-valued piecewise continuous functions on $[-h,0]$
$C^1([-h,0],R^n)$	Space of R^n-valued continuously differentiable functions on $[-h,0]$
$PC^1([-h,0],R^n)$	Space of R^n-valued piecewise continuously differentiable functions on $[-h,0]$
0_h	R^n-valued trivial function, $0_h(\theta) = 0 \in R^n$, $\theta \in [-h,0]$
$f(t+0)$	Right-hand-side limit of $f(t)$ at a point t, $f(t+0) = \lim_{\varepsilon\to 0} f(t+\|\varepsilon\|)$
$f(t-0)$	Left-hand-side limit of $f(t)$ at a point t, $f(t-0) = \lim_{\varepsilon\to 0} f(t-\|\varepsilon\|)$
$\|\varphi\|_h$	Uniform norm, $\|\varphi\|_h = \sup_{-h\le\theta\le 0} \|\varphi(\theta)\|$
$x'(t)$	First derivative of $x(t)$
$x''(t)$	Second derivative of $x(t)$
x_t	Restriction of $x(t)$, $x_t : \theta \to x(t+\theta)$, $\theta \in [-h,0]$
$Res\{f(s),s_0\}$	Residue of an analytical function $f(s)$ at a pole s_0
A^T	Transpose of a matrix A

A^*	Hermitian conjugate of a matrix A
$A > 0$ ($A \geq 0$)	Symmetric matrix A is positive definite (positive semidefinite)
$\lambda(A)$	Eigenvalue of a matrix A
$\lambda_{\max}(A)$, $\lambda_{\min}(A)$	Maximum, minimum eigenvalue of a symmetric matrix A
$\sigma(A)$	Spectrum of a square matrix A
$A \otimes B$	Kronecker product of matrices A and B
$vec(A)$	Vector of stacked columns of a matrix A

Chapter 1
General Theory

This chapter serves as a brief introduction to the theory of the retarded type time-delay system. It starts with a discussion of such basic notions as solutions, initial conditions, and the state of a time-delay system. Then some results on the existence and uniqueness of an initial value problem are presented. Continuity properties of the solutions are discussed as well. The main part of the chapter is devoted to stability analysis. Here we define concepts of stability, asymptotic stability, and exponential stability of the trivial solution of a time-delay system. Classical stability results, obtained using the Lyapunov–Krasovskii approach, are given in the form of necessary and sufficient conditions. A short section with historical comments concludes the chapter.

1.1 Preliminaries

We begin with a class of retarded type time-delay systems of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = g(t, x(t), x(t-h)), \tag{1.1}$$

where $x \in R^n$ and the time delay $h > 0$. Let the vector-valued function $g(t, x, y)$ be defined for $t \geq 0$, $x \in R^n$, and $y \in R^n$. We assume that this function is continuous in the variables.

1.1.1 Initial Value Problem

It is well known that a particular solution of a delay-free system, $\dot{x} = G(t, x)$, is defined by its initial conditions, which include an initial time instant t_0 and an initial

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state $x_0 \in R^n$. This is not the case when dealing with a solution of system (1.1). Here the knowledge of t_0 and x_0 is not sufficient even to define the value of the time derivative of $x(t)$ at the initial time instant t_0. To define a solution of system (1.1), one needs to select an initial time instant $t_0 \geq 0$ and an initial function $\varphi : [-h, 0] \to R^n$. The initial value problem for system (1.1) is formulated as follows. Given an initial time instant $t_0 \geq 0$ and an initial function φ, find a solution of the system that satisfies the condition

$$x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-h, 0]. \tag{1.2}$$

The initial function φ belongs to a certain functional space. It may be the space of continuous functions, $C([-h, 0], R^n)$, the space of piecewise continuous functions, $PC([-h, 0], R^n)$, or some other functional space. The choice of the space is dictated by a specific problem under investigation. In our case we assume that initial functions belong to the space $PC([-h, 0], R^n)$. Recall that the function φ belongs to the space if it admits at most a finite number of discontinuity points and for each continuity interval $(\alpha, \beta) \in [-h, 0]$ the function has a finite right-hand-side limit at $\theta = \alpha$, $\varphi(\alpha + 0) = \lim_{\varepsilon \to 0} \varphi(\alpha + |\varepsilon|)$, and a finite left-hand-side limit at $\theta = \beta$, $\varphi(\beta - 0) = \lim_{\varepsilon \to 0} \varphi(\beta - |\varepsilon|)$.

The Euclidean norm is used for vectors and the corresponding induced norm for matrices. The space $PC([-h, 0], R^n)$ is supplied with the standard uniform norm [24, 65, 66],

$$\|\varphi\|_h = \sup_{\theta \in [-h, 0]} \|\varphi(\theta)\|.$$

On the one hand, the fact that initial functions belong to a functional space gives rise to the interpretation of time-delay systems as a particular class of infinite-dimensional systems. On the other hand, the trajectories of a time-delay system lie in R^{n+1}; therefore, to some extent such systems can also be treated as systems in the finite-dimensional space.

1.1.2 Solutions

In this section we discuss the existence issue for the initial value problem (1.1)–(1.2). The approach presented here is known as the "step-by-step" method [3].

First, we consider a system on the segment $[t_0, t_0 + h]$. Here $t - h \in [t_0 - h, t_0]$, and $x(t - h)$ is defined by Eq. (1.2), $x(t - h) = \varphi(t - t_0 - h)$, and the system takes the form of the following auxiliary system of ordinary differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G^{(1)}(t, x) = g(t, x, \varphi(t - t_0 - h)), \qquad t \in [t_0, t_0 + h].$$

We are looking for a solution of the system that satisfies the condition $x(t_0) = \varphi(0)$.

If such a solution $\widetilde{x}(t)$ can be defined on the whole segment $[t_0, t_0 + h]$, then we address the next segment $[t_0 + h, t_0 + 2h]$. Here $t - h \in [t_0, t_0 + h]$, and the delay state $x(t-h)$ was already defined in the previous step, $x(t-h) = \widetilde{x}(t-h)$. Thus, on this segment system (1.1) is a delay-free system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G^{(2)}(t,x) = g(t,x,\widetilde{x}(t-h)), \qquad t \in [t_0 + h, t_0 + 2h],$$

and we are looking for a solution of the initial value problem $x(t_0 + h) = \widetilde{x}(t_0 + h)$.

Applying the step-by-step method, we reduce the computation of a solution of the initial value problem (1.1)–(1.2) to a series of standard initial value problems for a set of auxiliary systems of ordinary differential equations.

1.1.3 State Concept

In the theory of dynamic systems the concept of a system state occupies center stage. In general, we can say that the state of a system at a given time instant $t_1 \geq t_0$ should include the minimal information that allows one to continue the dynamic for $t \geq t_1$. If we adopt this point of view, then the state should be defined in the same manner as it was for the initial value problem.

The definition of the initial conditions and the step-by-step method of construction of the system solutions presented previously demonstrate that we need to know $x(t_1 + \theta)$, for $\theta \in [-h, 0]$, in order to continue a solution for $t \geq t_1$. Therefore, along a given solution of system (1.1) the state of the system at a time instant $t \geq t_0$ is defined as the restriction of the solution on the segment $[t-h, t]$. We use the following notation for the system state

$$x_t : \theta \to x(t+\theta), \quad \theta \in [-h, 0].$$

In the case where the initial condition (t_0, φ) should be indicated explicitly we use the notations $x(t, t_0, \varphi)$ and $x_t(t_0, \varphi)$. For time-invariant systems we usually assume that $t_0 = 0$ and omit the argument t_0 in these notations.

1.2 Existence and Uniqueness Issues

The dynamic of a time-delay system may depend not only on a delay state, $x(t-h)$, as happens in system (1.1), but on the complete state, x_t, of the system. An example of such a situation is given by the system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \int_{-h}^{0} g(t, x(t+\theta))\mathrm{d}\theta.$$

Here the right-hand side of the system depends on the values of $x(t+\theta)$, $\theta \in [-h,0]$. This means that the right-hand side is no longer a function but a functional that is defined on a particular functional space. It is clear that for such systems the step-by-step method is no longer applicable. Thus, we must look for an alternative procedure to compute solutions. Here we present such a procedure, but first we introduce a definition.

Definition 1.1 ([45]). Given a functional

$$F : PC([-h,0],R^n) \to R^n,$$

we say that the functional is continuous at a point $\varphi_0 \in PC([-h,0],R^n)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $\varphi \in PC([-h,0],R^n)$ the inequality $\|\varphi - \varphi_0\|_h < \delta$ implies that

$$\|F(\varphi) - F(\varphi_0)\| < \varepsilon.$$

Functional F is said to be continuous on a set $\Phi \subset PC([-h,0],R^n)$ if it is continuous at each point of the set.

Now we consider a functional

$$f : [0,\infty) \times PC([-h,0],R^n) \longrightarrow R^n.$$

The functional defines the time-delay system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t,x_t). \tag{1.3}$$

Theorem 1.1. *Given a time-delay system (1.3), where the functional*

$$f : [0,\infty) \times PC([-h,0],R^n) \longrightarrow R^n$$

satisfies the following conditions:

(i) For any $H > 0$ there exists $M(H) > 0$ such that

$$\|f(t,\varphi)\| \leq M(H), \quad (t,\varphi) \in [0,\infty) \times PC([-h,0],R^n), \text{ and } \|\varphi\|_h \leq H;$$

(ii) The functional $f(t,\varphi)$ is continuous on the set $[0,\infty) \times PC([-h,0],R^n)$ with respect to both arguments;

(iii) The functional $f(t,\varphi)$ satisfies the Lipschitz condition with respect to the second argument, i.e., for any $H > 0$ there exists a Lipschitz constant $L(H) > 0$ such that the inequality

$$\left\| f(t,\varphi^{(1)}) - f(t,\varphi^{(2)}) \right\| \leq L(H) \left\| \varphi^{(1)} - \varphi^{(2)} \right\|_h$$

holds for $t \geq 0$*,* $\varphi^{(k)} \in PC^1([-h,0],R^n)$*, and* $\left\|\varphi^{(k)}\right\|_h \leq H$*,* $k=1,2$*.*
Then, for a given $t_0 \geq 0$ *and an initial function* $\varphi \in PC([-h,0],R^n)$ *there exists* $\tau > 0$ *such that the system admits a unique solution* $x(t)$ *of the initial value problem (1.2), and the solution is defined on the segment* $[t_0-h, t_0+\tau]$*.*

Proof. Given $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, let us select $H > 0$ such that the inequality

$$H > H_0 = \|\varphi\|_h$$

holds. Now we can define the corresponding values $M = M(H)$ and $L = L(H)$.

Let us select $\tau > 0$ such that

$$\tau L < 1, \text{ and } \tau M < H - H_0,$$

and let us define a function $u : [t_0-h, t_0+\tau] \to R^n$ such that

$$u(t_0+\theta) = \varphi(\theta), \quad \theta \in [-h,0],$$

and the function is continuous on $[t_0, t_0+\tau]$. Assume additionally that the following inequality holds:

$$\|u(t) - \varphi(0)\| \leq (t-t_0)M, \quad t \in [t_0, t_0+\tau].$$

It follows from the definition that

$$\|u(t)\| \leq \|\varphi(0)\| + (t-t_0)M \leq \|\varphi\|_h + \tau M < H, \quad t \in [t_0, t_0+\tau].$$

We denote by U the set of all such functions. On the set U we define the operator $\mathcal{A}$ that acts on the functions of the set as follows:

$$\mathcal{A}(u)(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h, t_0], \\ \varphi(0) + \int\limits_{t_0}^{t} f(s, u_s)\mathrm{d}s, & t \in [t_0, t_0+\tau], \end{cases}$$

where $u_s : \theta \to u(s+\theta), \quad \theta \in [-h,0]$. It is a matter of simple calculation to check that the theorem conditions (i) and (ii) guarantee that the transformed function $\mathcal{A}(u)$ belongs to the same set U:

$$u \in U \Rightarrow \mathcal{A}(u) \in U.$$

Let $x(t, t_0, \varphi)$ be a solution of the initial value problem (1.3)–(1.2); then

$$x(t,t_0,\varphi) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h,t_0], \\ \varphi(0) + \int_{t_0}^{t} f(s,x_s(t_0,\varphi))\mathrm{d}s, & t \in [t_0,t_0+\tau], \end{cases}$$

and we conclude that this solution defines a fixed point of the operator $\mathcal{A}$.

Observe that

$$\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) = \begin{cases} 0, & t \in [t_0-h,t_0], \\ \int_{t_0}^{t} \left[f(s,u_s^{(1)}) - f(s,u_s^{(2)}) \right] \mathrm{d}s, & t \in [t_0,t_0+\tau]. \end{cases}$$

Hence for $t \in [t_0-h,t_0]$

$$\left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| = 0,$$

and for $t \in [t_0,t_0+\tau]$

$$\begin{aligned} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| &\le \left\| \int_{t_0}^{t} [f(s,u_s^{(1)}) - f(s,u_s^{(2)})]\mathrm{d}s \right\| \\ &\le \int_{t_0}^{t_0+\tau} \left\| f(s,u_s^{(1)}) - f(s,u_s^{(2)}) \right\| \mathrm{d}s. \end{aligned}$$

The Lipschitz condition (iii) implies that for $t \in [t_0,t_0+\tau]$

$$\begin{aligned} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| &\le \int_{t_0}^{t_0+\tau} L \left\| u_s^{(1)} - u_s^{(2)} \right\|_h \mathrm{d}s \\ &\le \tau L \sup_{s \in [t_0-h,t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\|. \end{aligned}$$

Since the preceding inequality holds for all $t \in [t_0-h,t_0+\tau]$, we conclude that

$$\sup_{s \in [t_0-h,t_0+\tau]} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| \le \tau L \sup_{s \in [t_0-h,t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\|.$$

Now, as $\tau L < 1$, the operator $\mathcal{A}$ satisfies the conditions of the contraction mapping theorem [45], and there exists a unique function $u^{(*)} \in U$ such that

$$u^{(*)}(t) = \mathcal{A}(u^{(*)})(t) = \begin{cases} \varphi(t - t_0), & t \in [t_0 - h, t_0], \\ \varphi(0) + \int\limits_{t_0}^{t} f(s, u_s^{(*)}) \mathrm{d}s, & t \in [t_0, t_0 + \tau]. \end{cases}$$

The functional $f(t,\varphi)$ is continuous, so differentiating the preceding equality,

$$\frac{\mathrm{d}u^{(*)}(t)}{\mathrm{d}t} = f(t, u_t^{(*)}), \quad t \in [t_0, t_0 + \tau],$$

we arrive at the conclusion that $u^{(*)}(t)$ is the unique solution of the initial value problem (1.3)–(1.2). □

Remark 1.1. We can take the new initial time instant, $t_1 = t_0 + \tau$, and define the new initial function

$$\varphi^{(1)}(\theta) = u^{(*)}(t_1 + \theta), \quad \theta \in [-h, 0].$$

Then the procedure can be repeated, and we extend the solution to the next segment $[t_1, t_1 + \widetilde{\tau}]$. This extension process can be continued as long as the solution remains bounded.

For each solution there exists a maximal interval $[t_0, t_0 + T)$ on which the solution is defined. Here we present conditions under which any solution of system (1.3) is defined on $[t_0, \infty)$.

Theorem 1.2. *Let system (1.3) satisfy the conditions of Theorem 1.1. Assume additionally that $f(t,\varphi)$ satisfies the inequality*

$$\|f(t,\varphi)\| \leq \eta(\|\varphi\|_h), \quad t \geq 0, \ \varphi \in PC([-h,0], R^n),$$

where the function $\eta(r)$, $r \in [0,\infty)$, is continuous, nondecreasing, and such that for any $r_0 \geq 0$ the following condition holds:

$$\lim_{R\to\infty} \int\limits_{r_0}^{R} \frac{\mathrm{d}r}{\eta(r)} = \infty.$$

Then any solution $x(t,t_0,\varphi)$ of the system is defined on $[t_0,\infty)$.

Proof. Given $t_0 \geq 0$ and $\varphi \in PC([-h,0], R^n)$, there exists a maximal interval $[t_0, t_0 + T)$ on which the corresponding solution $x(t,t_0,\varphi)$ is defined. For the sake of simplicity we denote $x(t,t_0,\varphi)$ by $x(t)$.

Assume by contradiction that $T < \infty$. Then there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \in [t_0, t_0 + T)$,

$$\lim_{k\to\infty} t_k = t_0 + T,$$

and

$$\lim_{k\to\infty} \|x(t_k)\| \to \infty;$$

otherwise, by Remark 1.1, the solution can be defined on a wider segment $[t_0, t_0 + T + \tau]$, where $\tau > 0$.

The solution satisfies the equality

$$x(t) = \varphi(0) + \int_{t_0}^{t} f(s, x_s)\mathrm{d}s, \quad t \in [t_0, t_0 + T).$$

It follows from the preceding equality and the theorem conditions that

$$\|x_t\|_h \le \|\varphi\|_h + \int_{t_0}^{t} \eta(\|x_s\|_h)\mathrm{d}s, \quad t \in [t_0, t_0 + T).$$

Denote the right-hand side of the last inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \eta(\|x_t\|_h) \le \eta(v(t)), \quad t \in [t_0, t_0 + T).$$

This implies that

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} \le t_k - t_0, \quad k = 1, 2, 3, \ldots.$$

On the one hand, as

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \int_{r_0}^{r_k} \frac{\mathrm{d}\xi}{\eta(\xi)},$$

where $r_0 = v(t_0) = \|\varphi\|_h \ge 0$, and

$$r_k = v(t_k) \ge \left\|x_{t_k}\right\|_h \ge \|x(t_k)\| \to \infty, \text{ as } k \to \infty,$$

then

$$\lim_{k\to\infty} \int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \infty.$$

On the other hand,

$$\lim_{k\to\infty} (t_k - t_0) = T.$$

Therefore, $T = \infty$, and we arrive at the contradiction with our assumption that $T < \infty$. This ends the proof of the statement. □

1.3 Continuity Properties

In this section we analyze the continuity properties of the solutions of system (1.3) with respect to the initial conditions and with respect to the system perturbations. These continuity properties are a direct consequence of the following theorem.

Theorem 1.3. *Assume that $f(t,\varphi)$ satisfies the conditions of Theorem 1.1. Let $x(t,t_0,\varphi)$ be a solution of system (1.3) such that*

$$x(t_0+\theta)=\varphi(\theta),\ \theta\in[-h,0].$$

Given the perturbed system

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}=f(t,y_t)+g(t,y_t),$$

where the functional $g(t,\varphi)$ is continuous on the set $[0,\infty)\times PC([-h,0],R^n)$, satisfies the Lipschitz condition with respect to the second argument, and

$$\|g(t,\varphi)\|\leq m,\quad t\geq 0,\ \varphi\in PC([-h,0],R^n),$$

let $y(t,t_0,\psi)$ be a solution of the perturbed system with the initial condition

$$y(t_0+\theta)=\psi(\theta),\ \theta\in[-h,0].$$

If the solutions are defined for $t\in[t_0-h,t_0+T]$, and if H is such that

$$\|x(t,t_0,\varphi)\|\leq H,\quad \|y(t,t_0,\psi)\|\leq H,\quad t\in[t_0-h,t_0+T],$$

then the inequality

$$\begin{aligned}\|x(t,t_0,\varphi)-y(t,t_0,\psi)\| &\leq \|x_t(t_0,\varphi)-y_t(t_0,\psi)\|_h\\ &\leq \left(\|\psi-\varphi\|_h+\frac{m}{L(H)}\right)\mathrm{e}^{L(H)(t-t_0)}\end{aligned}$$

holds for $t\in[t_0,t_0+T]$.

Proof. For the sake of simplicity we will use the following shorthand notations for the solutions $x(t)=x(t,t_0,\varphi)$ and $y(t)=y(t,t_0,\psi)$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-y(t)]=f(t,x_t)-f(t,y_t)-g(t,y_t),\quad t\in[t_0,t_0+T].$$

Integrating the preceding equality we obtain

$$x(t)-y(t)=\varphi(0)-\psi(0)+\int_{t_0}^{t}[f(s,x_s)-f(s,y_s)-g(s,y_s)]\mathrm{d}s, \quad t\in[t_0,t_0+T].$$

The last equality implies that for $t\in[t_0,t_0+T]$ the following inequalities hold:

$$\begin{aligned}\|x(t)-y(t)\| &\le \|\varphi(0)-\psi(0)\|+\int_{t_0}^{t}\|f(s,x_s)-f(s,y_s)-g(s,y_s)\|\,\mathrm{d}s\\ &\le \|\varphi(0)-\psi(0)\|+m\,(t-t_0)+L(H)\int_{t_0}^{t}\|x_s-y_s\|_h\,\mathrm{d}s.\end{aligned}$$

Since $\|\varphi(0)-\psi(0)\|\le\|\psi-\varphi\|_h$, we have

$$\begin{aligned}\|x(t)-y(t)\| &\le \|\varphi-\psi\|_h+m\,(t-t_0)\\ &\quad +L(H)\int_{t_0}^{t}\|x_s-y_s\|_h\,\mathrm{d}s, \quad t\in[t_0,t_0+T].\end{aligned}$$

Using similar arguments we can conclude that for $t_1\in[t-h,t]$, the inequality

$$\|x(t_1)-y(t_1)\|\le\|\varphi-\psi\|_h+m\,(t-t_0)+L(H)\int_{t_0}^{t}\|x_s-y_s\|_h\,\mathrm{d}s$$

holds, which implies

$$\sup_{t_1\in[t-h,t]}\|x(t_1)-y(t_1)\|\le\|\varphi-\psi\|_h+m\,(t-t_0)+L(H)\int_{t_0}^{t}\|x_s-y_s\|_h\,\mathrm{d}s.$$

So we have

$$\begin{aligned}\|x_t-y_t\|_h &\le \|\varphi-\psi\|_h+m\,(t-t_0)\\ &\quad +L(H)\int_{t_0}^{t}\|x_s-y_s\|_h\,\mathrm{d}s, \quad t\in[t_0,t_0+T].\end{aligned}$$

Denote the right-hand side of the preceding inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t}=m+L(H)\,\|x_t-y_t\|_h\le m+L(H)v(t), \quad t\in[t_0,t_0+T].$$

Integrating this inequality we arrive at the desired one:

$$\begin{aligned}\|x(t,t_0,\varphi)-y(t,t_0,\psi)\| &\le \|x_t(t_0,\varphi)-y_t(t_0,\psi)\|_h \\ &\le \|\psi-\varphi\|_h \mathrm{e}^{L(H)(t-t_0)}+\frac{m}{L(H)}(\mathrm{e}^{L(H)(t-t_0)}-1) \\ &\le \left(\|\psi-\varphi\|_h+\frac{m}{L(H)}\right)\mathrm{e}^{L(H)(t-t_0)}, \quad t\in[t_0,t_0+T].\end{aligned}$$

□

Corollary 1.1. *Let $g(t,\varphi)\equiv 0$; then $m=0$, and both $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ are solutions of system (1.3). Assume that these solutions are defined for $t\in[t_0,t_0+T]$. For any $\varepsilon>0$ there exists $\delta>0$ such that if $\|\psi-\varphi\|_h<\delta$, then the following inequality holds:*

$$\|x(t,t_0,\varphi)-x(t,t_0,\psi)\|<\varepsilon, \quad t\in[t_0,t_0+T].$$

In other words, $x(t,t_0,\varphi)$ depends continuously on φ.

Proof. The statement follows directly from Theorem 1.3 if we set $\delta=\varepsilon\mathrm{e}^{-L(H)T}$. □

Corollary 1.2. *Let $\psi(\theta)=\varphi(\theta)$, $\theta\in[-h,0]$; this means that the solutions $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ have the same initial conditions. Assume that these solutions are defined for $t\in[t_0,t_0+T]$. For any $\varepsilon>0$ there exists $\delta>0$ such that if $m<\delta$, then*

$$\|x(t,t_0,\varphi)-y(t,t_0,\varphi)\|<\varepsilon, \quad t\in[t_0,t_0+T].$$

This means that $x(t,t_0,\varphi)$ depends continuously on the right-hand side of system (1.3).

Proof. The statement follows directly from Theorem 1.3 if we set $\delta=L(H)\mathrm{e}^{-L(H)T}\varepsilon$. □

1.4 Stability Concepts

In this section we introduce some stability concepts for system (1.3). Let the system satisfy the conditions of Theorem 1.1. Assume additionally that the system admits a trivial solution, i.e., $f(t,0_h)\equiv 0$, for $t\ge 0$. Here 0_h stands for the trivial function, $0_h:\theta\to 0\in R^n$, $\theta\in[-h,0]$.

Definition 1.2 ([46]). The trivial solution of system (1.3) is said to be stable if for any $\varepsilon>0$ and $t_0\ge 0$ there exists $\delta(\varepsilon,t_0)>0$ such that for every initial function $\varphi\in PC([-h,0],R^n)$, $\|\varphi\|_h<\delta(\varepsilon,t_0)$, the following inequality holds:

$$\|x(t,t_0,\varphi)\|<\varepsilon, \quad t\ge t_0.$$

If $\delta(\varepsilon, t_0)$ can be chosen independently of t_0, then the trivial solution is said to be uniformly stable.

Remark 1.2. The value $\delta(\varepsilon, t_0)$ is always smaller than or equal to ε.

Proof. Assume that for some $\varepsilon > 0$ and $t_0 \geq 0$ we have $\delta(\varepsilon, t_0) > \varepsilon$; then there is $\varphi \in PC([-h, 0], R^n)$ such that $\|\varphi\|_h < \delta(\varepsilon, t_0)$, and $\|\varphi(0)\| > \varepsilon$. On the one hand, the corresponding solution $x(t, t_0, \varphi)$ should satisfy the inequality

$$\|x(t, t_0, \varphi)\| < \varepsilon, \quad t \geq t_0,$$

and, in particular, $\|x(t_0, t_0, \varphi)\| < \varepsilon$. On the other hand, $x(t_0, t_0, \varphi) = \varphi(0)$, so $\|x(t_0, t_0, \varphi)\| = \|\varphi(0)\| > \varepsilon$. This contradiction proves the remark. □

Definition 1.3. The trivial solution of system (1.3) is said to be asymptotically stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\Delta(\varepsilon, t_0) > 0$ such that for every initial function $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon, t_0)$, the following conditions hold.

1. $\|x(t, t_0, \varphi)\| < \varepsilon$, for $t \geq t_0$.
2. $x(t, t_0, \varphi) \to 0$ as $t - t_0 \longrightarrow \infty$.

   If $\Delta(\varepsilon, t_0)$ can be chosen independently of t_0 and there exists $H_1 > 0$ such that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \longrightarrow \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h \leq H_1$, then the trivial solution is said to be uniformly asymptotically stable.

Definition 1.4. The trivial solution of system (1.3) is said to be exponentially stable if there exist $\Delta_0 > 0$, $\sigma > 0$, and $\gamma \geq 1$ such that for every $t_0 \geq 0$ and any initial function $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \Delta_0$, the following inequality holds:

$$\|x(t, t_0, \varphi)\| \leq \gamma \|\varphi\|_h \mathrm{e}^{-\sigma(t-t_0)}, \quad t \geq t_0.$$

1.5 Lyapunov–Krasovskii Approach

First we show why the direct application of the classical Lyapunov approach does not work for time-delay systems. To this end, we consider a scalar linear equation of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t-h), \quad t \geq 0,$$

where a, b are real constants. Since the equation is linear, it seems natural to apply the positive-definite Lyapunov function $v(x) = x^2$. The time derivative of the function along the solutions of the equation is

$$\frac{\mathrm{d}v(x(t))}{\mathrm{d}t} = 2x(t)[ax(t) + bx(t-h)] = 2ax^2(t) + 2bx(t)x(t-h).$$

For the case $b = 0$ the equation is delay free, and the time derivative is negative definite when $a < 0$. According to the Lyapunov stability theory, this implies the asymptotic stability of the equation.

The situation becomes different when $b \neq 0$. In this case the time derivative includes two terms and, despite the fact that the first term remains negative definite for $a < 0$, we are not able to state the same about the time derivative because nothing certain can be said about the sign and the value of the second term, $2bx(t)x(t-h)$. Therefore, some modifications of the Lyapunov approach should be made if we would like to apply it to a stability analysis of time-delay systems.

Such modifications have been proposed in two distinct ways.

1. The first one is due to N. N. Krasovskii, who proposed to replace classical Lyapunov functions that depend on the instant state, $x(t)$, of a system by functionals that depend on the true state, x_t. This modification is now known as the Lyapunov–Krasovskii approach [46–48].
2. The other modification was proposed by Razumikhin [61,62]. It uses the classical Lyapunov functions but adds an additional condition that allows one to compare the values of $x(t)$ and $x(t-h)$ and provides negativity conditions for the time derivative of the functions along the solutions of the system.

In this book we do not treat the Razumikhin approach but concentrate on the Lyapunov–Krasovskii one. We start with the definition of positive-definite functions.

Definition 1.5. A function $v_1(x)$ is said to be positive definite if there exists $H > 0$ such that the function is continuous on the set $\{x \mid \|x\| \leq H\}$ and satisfies the following conditions:

1. $v_1(0) = 0$;
2. $v_1(x) > 0$ for $0 < \|x\| \leq H$.

Now we extend the positive-definiteness concept to the case of functionals.

Definition 1.6. Functional $v(t,\varphi)$ is said to be positive definite if there exists $H > 0$ such that the following conditions are satisfied.

1. The functional $v(t,\varphi)$ is defined for $t \geq 0$ and any $\varphi \in PC([-h,0],R^n)$ with $\|\varphi\|_h \leq H$.
2. $v(t,0_h) = 0,\ t \geq 0$.
3. There exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \leq v(t,\varphi),\ \ t \geq 0, \text{ and } \varphi \in PC([-h,0],R^n),\ \ \|\varphi\|_h \leq H.$$

4. For any given $t_0 \geq 0$ the functional $v(t_0,\varphi)$ is continuous in φ at the point 0_h, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|\varphi\|_h < \delta$ implies

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) < \varepsilon.$$

We are now ready to present some basic statements of the Lyapunov–Krasovskii approach.

Theorem 1.4. *The trivial solution of system (1.3) is stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that along the solutions of the system the value of the functional $v(t,x_t)$ as a function of t does not increase.*

Proof. Sufficiency: The positive definiteness of the functional $v(t,\varphi)$ implies that there exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \leq v(t,\varphi), \ \ t \geq 0, \text{ and } \varphi \in PC([-h,0],R^n), \ \ \|\varphi\|_h \leq H.$$

For a given $\varepsilon > 0$ $(\varepsilon < H)$ we define the positive value

$$\lambda(\varepsilon) = \min_{\|x\|=\varepsilon} v_1(x). \tag{1.4}$$

Since for a given $t_0 \geq 0$ the functional $v(t_0,\varphi)$ is continuous in φ at the point 0_h, there exists $\delta > 0$ such that $v(t_0,\varphi) < \lambda(\varepsilon)$ for any $\varphi \in PC([-h,0],R^n)$ with $\|\varphi\|_h < \delta$.

It is clear that $\delta \leq \varepsilon$; otherwise we could present an initial function $\varphi \in PC([-h,0],R^n)$ such that $\|\varphi\|_h < \delta$ and $\|\varphi(0)\| = \varepsilon$. On the one hand, for this initial function we have $v_1(\varphi(0)) \geq \lambda(\varepsilon)$. On the other hand, $v_1(\varphi(0)) \leq v(t_0,\varphi) < \lambda(\varepsilon)$. The contradiction proves the inequality $\delta \leq \varepsilon$.

Now let $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h < \delta$. Then the theorem condition implies that

$$v_1(x(t,t_0,\varphi)) \leq v(t,x_t(t_0,\varphi)) \leq v(t_0,\varphi) < \lambda(\varepsilon), \quad t \geq t_0. \tag{1.5}$$

Assume by contradiction that there exists a time instant $t_1 \geq t_0$ for which $\|x(t_1,t_0,\varphi)\| \geq \varepsilon$. Since for $t \geq t_0$ the function $\|x(t,t_0,\varphi)\|$ is continuous in t, and since $\|x(t_0,t_0,\varphi)\| = \|\varphi(0)\| \leq \|\varphi\|_h < \delta \leq \varepsilon$, there exists $t^* \in [t_0,t_1]$ such that $\|x(t^*,t_0,\varphi)\| = \varepsilon$. So, on the one hand, by Eq. (1.4), we know that

$$v_1(x(t^*,t_0,\varphi)) \geq \lambda(\varepsilon).$$

On the other hand, inequality (1.5) implies the inequality

$$v_1(x(t^*,t_0,\varphi)) < \lambda(\varepsilon),$$

which contradicts the previous one. The contradiction proves that our assumption is wrong, and the following inequality holds:

$$\|x(t,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

This means that δ satisfies Definition 1.2, and therefore the trivial solution of system (1.3) is stable.

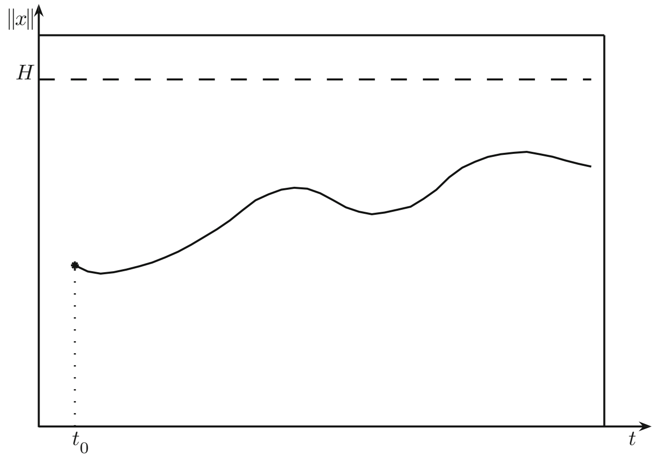

Fig. 1.1 Value of $\|x(t,t_0,\varphi\|$, the first case

Necessity: Now, the trivial solution of system (1.3) is stable, and we must prove that there exists a functional $v(t,\varphi)$ that satisfies the theorem condition.

Construction of the functional: Since the trivial solution of system (1.3) is stable, for $\varepsilon = H$ there exists $\delta(H,t_0) > 0$ such that the inequality $\|\varphi\|_h < \delta(H,t_0)$ implies that $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. We define the functional $v(t,\varphi)$ as follows:

$$v(t_0,\varphi) = \begin{cases} \sup\limits_{t \geq t_0} \|x(t,t_0,\varphi)\|, \text{ if } \|x(t,t_0,\varphi)\| < H, \text{ for } t \geq t_0, \\ H, \text{ if there exists } T \geq t_0 \text{ such that } \|x(T,t_0,\varphi)\| = H. \end{cases} \tag{1.6}$$

These two possibilities are illustrated in Figs. 1.1 and 1.2, respectively.

We check first that the functional $v(t,\varphi)$ is positive definite. To this end, we must verify that it satisfies the conditions of Definition 1.6.

Condition 1: The value $v(t_0,\varphi)$ is defined for all $t_0 \geq 0$, and every initial function $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.

Condition 2: For the trivial initial function, $\varphi = 0_h$, the corresponding solution is trivial, $x(t,t_0,0_h) = 0$, for $t \geq t_0$. Thus $v(t_0,0_h) = 0$, $t_0 \geq 0$.

Condition 3: The function $v_1(x) = \|x\|$ is positive definite. Given $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H$, in the case where $\|x(t,t_0,\varphi)\| < H$, for $t \geq t_0$, we have

$$v_1(\varphi(0)) = \|\varphi(0)\| \leq \sup_{t \geq t_0} \|x(t,t_0,\varphi)\| = v(t_0,\varphi).$$

In the other case, where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| = H$, we have

$$v_1(\varphi(0)) = \|\varphi(0)\| \leq H = v(t_0,\varphi).$$

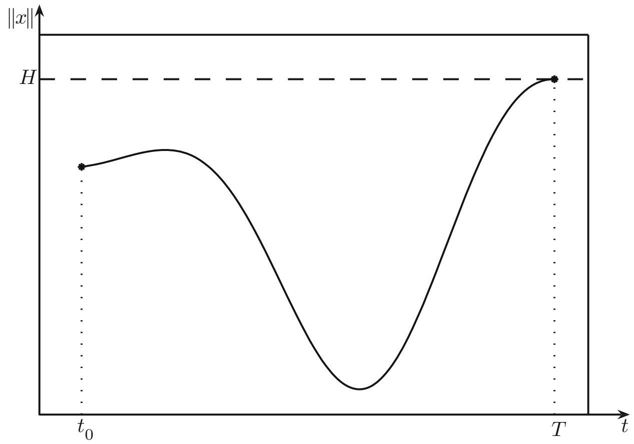

Fig. 1.2 Value of $\|x(t,t_0,\varphi\|$, the second case

Condition 4: Given $t_0 \geq 0$, the stability of the trivial solution means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\varphi\|_h < \delta$ implies $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. In other words, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\varphi\|_h < \delta$ implies

$$v(t_0,\varphi) = |v(t_0,\varphi) - v(t_0,0_h)| \leq \varepsilon.$$

This observation makes it clear that for a fixed $t_0 \geq 0$ the functional $v(t_0,\varphi)$ is continuous in φ at the point 0_h.

Now we check that functional (1.6) satisfies the theorem condition. First, we consider the case where $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. In this case, given two time instants, t_1 and t_2, such that $t_2 > t_1 \geq t_0$, we compare the values

$$v(t_1,x_{t_1}(t_0,\varphi)) = \sup_{t\geq t_1} \|x(t,t_0,\varphi)\|$$

and

$$v(t_2,x_{t_2}(t_0,\varphi)) = \sup_{t\geq t_2} \|x(t,t_0,\varphi)\|\,.$$

Since for the second value the range of the supremum is smaller than that for the first value, we conclude that

$$v(t_2,x_{t_2}(t_0,\varphi)) \leq v(t_1,x_{t_1}(t_0,\varphi)).$$

This means that the functional $v(t,x_t(t_0,\varphi))$ does not increase along the solution. In the second case, where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| = H$, we have the equality

$$v(t_2,x_{t_2}(t_0,\varphi)) = v(t_1,x_{t_1}(t_0,\varphi)) = H,$$

and, once again, the functional does not increase along the solution of system (1.3). □

Remark 1.3. The functional $v(t,\varphi)$, defined in the proof of the necessity part of Theorem 1.4, is of academic interest only. Obviously, we cannot use such functionals in applications. The computation of practically useful Lyapunov functionals is not a simple task.

Theorem 1.5. *The trivial solution of system (1.3) is uniformly stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions are satisfied.*

1. *The value of the functional along the solutions of the system, $v(t,x_t)$, does not increase.*
2. *The functional is continuous in φ at the point 0_h, uniformly for $t \geq 0$.*

Proof. Sufficiency: In the proof of the sufficiency part of Theorem 1.4 the value $\delta = \delta(\varepsilon,t_0)$ was chosen such that for any $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h < \delta$, the value of the functional for a given $t_0 \geq 0$ satisfies the inequality $v(t_0,\varphi) < \lambda(\varepsilon)$. Since now the functional is continuous in φ at the point 0_h, uniformly for $t \geq 0$, the value δ can be chosen independently of t_0.

Necessity: The uniform stability of the trivial solution of system (1.3) implies that δ can be chosen independently of t_0, $\delta = \delta(\varepsilon)$. It was demonstrated in the proof of Theorem 1.4 that functional (1.6) is positive definite and does not increase along the solutions of system (1.3). We show that this functional satisfies the second condition of the theorem. For any $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h < \delta(\varepsilon)$, and any $t_0 \geq 0$ we have that $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. This means that

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) \leq \varepsilon.$$

In other words, functional (1.6) is continuous in φ at the point 0_h, uniformly for $t \geq 0$. □

Remark 1.4. The second condition of Theorem 1.5 is satisfied when $v(t,\varphi)$ admits an upper estimate of the form

$$v(t,\varphi) \leq v_2(\varphi), \quad t \geq 0,\ \varphi \in PC([-h,0],R^n),\ \|\varphi\|_h \leq H,$$

with a positive-definite functional $v_2(\varphi)$.

Theorem 1.6. *The trivial solution of system (1.3) is asymptotically stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions hold.*

1. *The value of the functional along the solutions of the system, $v(t,x_t)$, does not increase.*
2. *For any $t_0 \geq 0$ there exists a positive value $\mu(t_0)$ such that if $\varphi \in PC([-h,0],R^n)$ and $\|\varphi\|_h < \mu(t_0)$, then $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t - t_0 \to \infty$.*

Proof. Sufficiency: The first condition of the theorem implies the stability of the trivial solution of system (1.3) (Theorem 1.4). Thus, for any $\varepsilon > 0$ $(\varepsilon < H)$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that if $\|\varphi\|_h < \delta(\varepsilon, t_0)$, then $\|x(t, t_0, \varphi)\| < \varepsilon$ for $t \geq t_0$. Let us define the value

$$\Delta(\varepsilon, t_0) = \min\{\delta(\varepsilon, t_0), \mu(t_0)\}.$$

For any given initial function $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon, t_0)$, the following inequality holds:

$$\|x(t, t_0, \varphi)\| < \varepsilon, \quad t \geq t_0.$$

We will demonstrate that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$. The functional $v(t, \varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \leq v(t, \varphi), \text{ for } t \geq 0, \text{ and } \varphi \in PC([-h, 0], R^n), \|\varphi\|_h \leq H.$$

The function $v_1(x)$ is continuous, so for any given $\varepsilon_1 > 0$ $(\varepsilon_1 < \varepsilon)$ we may define the positive value

$$\alpha = \min_{\varepsilon_1 \leq \|x\| \leq \varepsilon} v_1(x).$$

By the second condition of the theorem, there exists $T > 0$ such that $v(t, x_t(t_0, \varphi)) < \alpha$ for $t \geq t_0 + T$. This implies the inequality

$$v_1(x(t, t_0, \varphi)) < \alpha, \quad t \geq t_0 + T,$$

and we conclude that

$$\|x(t, t_0, \varphi)\| < \varepsilon_1, \quad t \geq t_0 + T.$$

This means that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$, and we must accept that the previously defined value $\Delta(t_0, \varepsilon)$ satisfies Definition 1.3.

Necessity: In this part of the proof we make use of functional (1.6). In the proof of Theorem 1.4 it was demonstrated that the functional is positive definite and does not increase along the solutions of system (1.3). This means that the functional satisfies the first condition of the theorem.

We address the second condition of the theorem and choose the value $\mu(t_0)$ as follows:

$$\mu(t_0) = \Delta(H, t_0) > 0.$$

Now, for any initial function $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \mu(t_0)$, we know that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$. This means that for any $\varepsilon_1 > 0$ there exists $T > 0$ such that $\|x(t, t_0, \varphi)\| < \varepsilon_1$ for $t \geq t_0 + T$. According to Eq. (1.6), we have

$$v(t, x_t(t_0, \varphi)) = \sup_{s \geq t} \|x(s, t_0, \varphi)\| \leq \varepsilon_1, \text{ for } t \geq t_0 + T.$$

The preceding observation means that $v(t,x_t(t_0,\varphi))$ tends to zero as $t\to\infty$. □

Theorem 1.7. *The trivial solution of system (1.3) is uniformly asymptotically stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions hold.*

1. *The value of the functional along the solutions of the system, $v(t,x_t)$, does not increase.*
2. *The functional is continuous in φ at the point 0_h, uniformly for $t\geq 0$.*
3. *There exists a positive value μ_1 such that $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t-t_0\to\infty$, uniformly with respect to $t_0\geq 0$, and $\varphi\in PC([-h,0],R^n)$, with $\|\varphi\|_h\leq\mu_1$.*

Proof. Sufficiency: Comparing this theorem with Theorem 1.5 we conclude that the trivial solution of system (1.3) is uniformly stable. Therefore, for a given $\varepsilon>0$ the value

$$\Delta(\varepsilon)=\min\{\mu_1,\delta(\varepsilon)\}>0$$

is such that the following properties hold:

1. Given $t_0\geq 0$ and $\varphi\in PC([-h,0],R^n)$, with $\|\varphi\|_h<\Delta(\varepsilon)$, then $\|x(t,t_0,\varphi)\|<\varepsilon$ for $t\geq t_0$.
2. $v(t,x_t(t_0,\varphi))\to 0$ as $t-t_0\to\infty$.

   Now we define

$$H_1=\frac{1}{2}\Delta(H)>0.$$

The functional $v(t,\varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0))\leq v(t,\varphi),\ \text{for } t\geq 0, \text{ and } \varphi\in PC([-h,0],R^n),\ \|\varphi\|_h\leq H.$$

The function $v_1(x)$ is continuous; therefore, for any $\varepsilon_1>0$ $(\varepsilon_1<H)$ we may define the positive value

$$\alpha=\min_{\varepsilon_1\leq\|x\|\leq H}v_1(x).$$

By the third condition of the theorem there exists $T>0$ such that for any $t_0\geq 0$ and $\varphi\in PC([-h,0],R^n)$, with $\|\varphi\|_h\leq H_1$, the following inequality holds:

$$v(t,x_t(t_0,\varphi))<\alpha,\ t-t_0\geq T.$$

This implies that

$$v_1(x(t,t_0,\varphi))<\alpha,\quad t-t_0\geq T,$$

and we conclude that

$$\|x(t,t_0,\varphi)\|<\varepsilon_1,\quad t-t_0\geq T,$$

for any $t_0 \geq 0$, and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$. Therefore, the previously defined values $\Delta(\varepsilon)$ and H_1 satisfy Definition 1.3. This ends the proof of the sufficiency part of the theorem.

Necessity: The uniform asymptotic stability of the trivial solution of system (1.3) implies that functional (1.6) satisfies the first two conditions of the theorem. Let us set

$$\mu_1 = \frac{1}{2}\Delta(H),$$

where $\Delta(\varepsilon)$ is from Definition 1.3. Now, given $\varepsilon_1 > 0$, for any $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$, there exists $T > 0$ such that

$$\|x(t,t_0,\varphi)\| < \varepsilon_1, \quad t - t_0 \geq T.$$

This means that functional (1.6) satisfies the inequality

$$v(t,x_t(t_0,\varphi)) = \sup_{s\geq 0} \|x(s,t_0,\varphi)\| \leq \varepsilon_1, \quad t - t_0 \geq T,$$

i.e., the functional decreases monotonically to zero as $t - t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$. This ends the proof of the necessity part. □

The following statement provides sufficient conditions of the uniform asymptotic stability of the trivial solution of system (1.3).

Theorem 1.8 ([46]). *The trivial solution of system (1.3) is uniformly asymptotically stable if there exist two positive-definite functionals, $v(t,\varphi)$ and $v_2(\varphi)$, and a positive-definite function $w(x)$ such that the following two conditions hold.*

1. *$v(t,\varphi) \leq v_2(\varphi)$, for $t \geq 0$, and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.*
2. *The value of the functional along the solutions of the system is differentiable by t, and its time derivative satisfies the inequality*

$$\frac{\mathrm{d}v(t,x_t)}{\mathrm{d}t} \leq -w(x(t)).$$

Proof. Observe that the first condition of the theorem implies that the functional $v(t,\varphi)$ is continuous in φ at the point 0_h, uniformly for $t \geq 0$ (Corollary 1.4). This means that the second condition of Theorem 1.7 is satisfied. The first condition of Theorem 1.7 follows directly from the second condition of this theorem.

Now we show that the third condition of Theorem 1.7 is also satisfied. It is evident that the theorem conditions guarantee that the trivial solution is uniformly stable, i.e., for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ that satisfies the definition of the uniform stability. The functional $v_2(\varphi)$ is positive definite, so there exists a positive value η such that the following inequality holds:

$$v_2(\varphi) < H, \quad \varphi \in PC([-h,0], R^n), \text{ with } \|\varphi\|_h \leq \eta.$$

Let us set

$$\mu_1 = \min\left\{\frac{1}{2}\delta(H), \eta\right\}.$$

We are going to demonstrate that for any given $\alpha > 0$ there exists $T > 0$ such that if $t - t_0 \geq T$, then the inequality

$$v(t, x_t(t_0, \varphi)) < \alpha$$

holds for any $t_0 \geq 0$ and $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h \leq \mu_1$. Since the functional $v_2(\varphi)$ is positive definite, there exists $\beta > 0$ such that the inequality $\|\varphi\|_h < \beta$ implies $v_2(\varphi) < \alpha$. The function $w(x)$ is positive definite, and we can define a positive constant γ as follows:

$$\gamma = \min_{\frac{\beta}{2} \leq \|x\| \leq H} w(x).$$

For any function $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h \leq H$, we have

$$\|f(t,\varphi)\| \leq M(H), \text{ for } t \geq 0.$$

Now we set

$$\tau = \min\left\{h, \frac{\beta}{2M(H)}\right\}$$

and select an entire number N satisfying the inequality

$$H - \gamma\tau N < 0.$$

Finally, we define a positive value T as follows:

$$T = 2hN.$$

Given an initial instant $t_0 \geq 0$ and function $\varphi \in PC([-h,0], R^n)$ such that $\|\varphi\|_h \leq \mu_1$, we will demonstrate that $v(t, x_t(t_0,\varphi)) < \alpha$ for $t - t_0 \geq T$. First we observe that the second condition of the theorem implies that $v(t, x_t(t_0,\varphi))$ is a decreasing function of t, so it is enough to check that $v\left(t_0 + T, x_{t_0+T}(t_0,\varphi)\right) < \alpha$. Assume by contradiction that this is not the case, and $v\left(t_0 + T, x_{t_0+T}(t_0,\varphi)\right) \geq \alpha$. This means that

$$\alpha \leq v(t, x_t(t_0,\varphi)) \leq v_2(x_t(t_0,\varphi))$$

for $t \in [t_0, t_0 + T]$. The inequality $\alpha \leq v_2(x_t(t_0,\varphi))$ implies that $\|x_t(t_0,\varphi)\|_h \geq \beta$ for $t \in [t_0, t_0 + T]$, i.e., in each segment $[t - h, t] \subset [t_0, t_0 + T]$ there exists a point $t^* \in [t - h, t]$ such that $\|x(t^*, t_0, \varphi)\| \geq \beta$. These arguments demonstrate that we can define an increasing sequence, $\{t_j\}_{j=1}^{N}$, such that at the points of the sequence $\left\|x(t_j,t_0,\varphi)\right\| \geq \beta$. Without any loss of generality we assume that any two consecutive points of the sequence satisfy the inequalities $h < t_{j+1} - t_j < 2h$.

According to the choice of the initial function φ, we know that $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$, and at the points of the sequence the following inequality holds:

$$\beta \leq \left\|x(t_j,t_0,\varphi)\right\|, \quad j = 1,2,\ldots,N.$$

Now observe that

$$x(t,t_0,\varphi) = x(t_j,t_0,\varphi) + \int_{t_j}^{t} f(s,x_s(t_0,\varphi))\mathrm{d}s, \quad t \geq t_j,$$

and, since $\|f(s,x_s(t_0,\varphi))\| \leq M(H)$, for $t \geq 0$ we have

$$\begin{aligned}\left\|x(t,t_0,\varphi) - x(t_j,t_0,\varphi)\right\| &\leq \int_{t_j}^{t} \|f(s,x_s(t_0,\varphi))\|\,\mathrm{d}s \\ &\leq \tau M(H), \text{ for } t \in [t_j,t_j+\tau].\end{aligned}$$

According to our choice of τ, we conclude that for $t \in [t_j,t_j+\tau]$

$$\left\|x(t,t_0,\varphi) - x(t_j,t_0,\varphi)\right\| \leq \frac{\beta}{2}.$$

As $\left\|x(t_j,t_0,\varphi)\right\| \geq \beta$, the inequality

$$\|x(t,t_0,\varphi)\| \geq \frac{\beta}{2}, \quad t \in [t_j,t_j+\tau],$$

holds for $j = 1,2,\ldots,N$. It is evident that

$$w(x(t,t_0,\varphi)) \geq \gamma,\ t \in [t_j,t_j+\tau], \quad j = 1,2,\ldots,N,$$

and the second condition of the theorem implies that

$$\begin{aligned}v(t_0+T,x_{t_0+T}(t_0,\varphi)) &\leq v(t_0,\varphi) - \int_{t_0}^{t_0+T} w(x(s,t_0,\varphi)\mathrm{d}s \\ &\leq H - \gamma\tau N < 0.\end{aligned}$$

This means that $v(t_0+T,x_{t_0+T}(t_0,\varphi))$ is negative, which contradicts the positive definiteness of the functional $v(t,\varphi)$. The contradiction proves that

$$v(t, x_t(t_0, \varphi)) < \alpha, \text{ for } t - t_0 \geq T.$$

Now to end the proof, it is enough to refer to Theorem 1.7. □

Theorem 1.9. *The trivial solution of system (1.3) is exponentially stable if there exists a positive-definite functional* $v(t, \varphi)$ *such that the following conditions hold.*

1. *There are two positive constants* α_1, α_2 *for which*

$$\alpha_1 \|\varphi(0)\|^2 \leq v(t, \varphi) \leq \alpha_2 \|\varphi\|_h^2, \text{ for } t \geq 0,$$

   for $t \geq 0$, *and* $\varphi \in PC([-h, 0], R^n)$ *with* $\|\varphi\|_h \leq H$.
2. *The functional is differentiable along the solutions of the system, and there exists a positive constant* σ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t, x_t) + 2\sigma v(t, x_t) \leq 0.$$

Proof. Let us define the positive-definite function $v_1(x) = \alpha_1 \|x\|^2$ and the positive-definite functional $v_2(\varphi) = \alpha_1 \|\varphi\|_h^2$. It is evident that the functional $v(t, \varphi)$ satisfies the conditions of Theorem 1.5. Therefore, the trivial solution of system (1.3) is uniformly stable, and for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the inequality $\|\varphi\|_h < \delta(\varepsilon)$ implies $\|x(t, t_0, \varphi)\| < \varepsilon$ for $t \geq t_0$. We will show that the value $\Delta_0 = \delta(H)$ satisfies Definition 1.4. To this end, assume that $t_0 \geq 0$ and $\varphi \in PC([-h, 0], R^n)$, $\|\varphi\|_h < \Delta_0$. The corresponding solution $x(t, t_0, \varphi)$ is such that

$$\|x(t, t_0, \varphi)\| < H, \text{ for } t \geq t_0.$$

The second condition of the theorem implies the inequality

$$v(t, x_t(t_0, \varphi)) \leq v(t_0, \varphi) \mathrm{e}^{-2\sigma(t - t_0)}, \quad t \geq t_0.$$

Applying the first condition of the theorem we obtain that

$$\alpha_1 \|x(t, t_0, \varphi)\|^2 \leq v(t_0, \varphi) \mathrm{e}^{-2\sigma(t - t_0)} \leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma(t - t_0)}, \quad t \geq t_0.$$

The preceding inequalities provide the desired exponential estimate

$$\|x(t, t_0, \varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|\varphi\|_h \mathrm{e}^{-\sigma(t - t_0)}, \quad t \geq t_0.$$

□

1.6 Notes and References

The origins of the time-delay systems go back to such giants as L. Euler, J. L. Lagrange, and P. Laplace. A systematic development of the theory of functional differential equations began in the twentieth century with Volterra [69, 70], Myshkis [57], Krasovskii [46], Bellman and Cooke [3], Halanay [19], and Hale [21], to mention just the principal contributors.

The restriction of a solution, $x_t : \theta \to x(t+\theta)$, $\theta \in [-h, 0]$, as the true state of a time-delay system was introduced by Krasovskii [48]. This allowed him to develop the stability theory of time-delay systems to the same level as that of ordinary differential equations [46].

In the exposition of the basic existence and continuity results we follow the excellent book by Halanay [19]; see also [3, 6, 10, 11, 20, 23, 49].

The foundations of the Lyapunov second approach for time-delay systems, which is now known as the Lyapunov–Krasovskii approach, were developed by Krasovskii [46–48]; see also [44, 58]. The form of presentation of the stability results in Sect. 1.5 was inspired by Zubov [72].

Chapter 2
Single Delay Case

In this chapter we consider the class of retarded type linear systems with one delay. There are several reasons for restricting our attention to this class before proceeding to more general ones. First, from a methodological point of view, it seems that dealing with single-delay systems simplifies the understanding of basic concepts and creates a firm basis for developing the concepts for more general cases. Second, for the case of single delay we often obtain more complete results than in a more general setting. Finally, results for the single-delay case are not as cumbersome as those for the more general classes of time-delay systems.

We introduce the fundamental matrix of such a system and provide an explicit expression for the solution of an initial value problem. Exponential stability conditions, both in terms of characteristic eigenvalues of the system and in terms of Lyapunov functionals, are presented. The general scheme for the computation of quadratic functionals with prescribed time derivatives along the solutions of a time-delay system is explained in detail. It is demonstrated that the functionals are defined by special matrix-valued functions. We show that these matrix-valued functions are natural counterparts of the classical Lyapunov matrices that appear in the computation of Lyapunov quadratic forms for a delay-free linear system; therefore, they are known as Lyapunov matrices for a time-delay system. A substantial part of the chapter is devoted to an analysis of the basic properties of the Lyapunov matrices. Then, Lyapunov functionals that admit various quadratic lower and upper bounds are introduced. They are called functionals of the complete type. Finally, we make use of complete type functionals to derive exponential estimates of the solutions of time-delay systems, robustness bounds for perturbed systems, evaluation of quadratic performance indices, and computation of critical values of system parameters.

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2.1 Preliminaries

We consider a retarded type time-delay system of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0x(t) + A_1x(t-h), \quad t \geq 0. \tag{2.1}$$

Here A_0, A_1 are given real $n\times n$ matrices, and h is a positive time delay.

Let $\varphi : [-h,0] \to R^n$ be an initial function. We assume that the function belongs to the space, $PC([-h,0],R^n)$, of piecewise continuous functions defined on the segment $[-h,0]$. Let $x(t,\varphi)$ stand for the solution of system (2.1) under the initial condition

$$x(\theta,\varphi) = \varphi(\theta), \quad \theta \in [-h,0],$$

and let $x_t(\varphi)$ denote the restriction of the solution to the segment $[t-h,t]$

$$x_t(\varphi) : \theta \to x(t+\theta,\varphi), \quad \theta \in [-h,0].$$

We omit the argument φ in these notations and write $x(t)$ and x_t instead of $x(t,\varphi)$ and $x_t(\varphi)$ where no confusion may arise.

Recall that the Euclidean norm is used for vectors and the induced matrix norm for matrices. For elements of the space $PC([-h,0],R^n)$ we use the uniform norm

$$\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|.$$

The principal objective of this section is to present an explicit expression for the solutions of system (2.1) in terms of their initial functions.

2.1.1 Fundamental Matrix

The key element needed to derive this expression is the fundamental matrix of the system. In many respects, the fundamental matrix plays the same role for system (2.1) as the matrix exponent does for linear delay-free systems.

Definition 2.1 ([3]). It is said that the $n\times n$ matrix $K(t)$ is the fundamental matrix of system (2.1) if it satisfies the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = K(t)A_0 + K(t-h)A_1, \quad t \geq 0, \tag{2.2}$$

and $K(t) = 0_{n\times n}$ for $t < 0$, $K(0) = I$. Here I is the identity matrix.

Remark 2.1. The fundamental matrix also satisfies the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = A_0K(t) + A_1K(t-h), \quad t \geq 0.$$

This does not mean that matrix $K(t)$ commutes individually with the coefficient matrices A_k, $k = 0,1$.

Proof. To verify this remark, it is sufficient to compare the Laplace image of the fundamental matrix as a solution of matrix Eq. (2.2) with that of the matrix equation given in the remark. □

2.1.2 Cauchy Formula

Now we are ready to present the main result of this section.

Theorem 2.1 ([3]). *Given an initial function $\varphi \in PC([-h,0],R^n)$, the following equality holds:*

$$x(t,\varphi) = K(t)\varphi(0) + \int_{-h}^{0} K(t-\theta-h)A_1\varphi(\theta)\mathrm{d}\theta, \quad t \geq 0. \tag{2.3}$$

This expression for $x(t,\varphi)$ is known as the Cauchy formula.

Proof. Let $t > 0$ and assume that $\xi \in (0,t)$; then

$$\begin{aligned}\frac{\partial}{\partial \xi}\left[K(t-\xi)x(\xi,\varphi)\right] = &-\left[K(t-\xi)A_0 + K(t-\xi-h)A_1\right]x(\xi,\varphi)\\ &+K(t-\xi)\left[A_0x(\xi,\varphi) + A_1x(\xi-h,\varphi)\right].\end{aligned}$$

Integrating the last equality by ξ from 0 to t we obtain that

$$x(t,\varphi) - K(t)\varphi(0) = -\int_0^t K(t-\xi-h)A_1x(\xi,\varphi)\mathrm{d}\xi + \int_0^t K(t-\xi)A_1x(\xi-h,\varphi)\mathrm{d}\xi.$$

The second integral on the right-hand side of the preceding equality can be transformed as follows:

$$\int_0^t K(t-\xi)A_1x(\xi-h,\varphi)\mathrm{d}\xi = \langle \theta = \xi - h \rangle = \int_{-h}^{t-h} K(t-\theta-h)A_1x(\theta,\varphi)\mathrm{d}\theta.$$

Since the matrix $K(\theta) = 0_{n\times n}$ for $\theta \in [-h, 0)$, the upper limit of the integral on the right-hand side can be increased up to t,

$$\int_{-h}^{t-h} K(t-\theta-h)A_1x(\theta,\varphi)\mathrm{d}\theta = \int_{-h}^{t} K(t-\theta-h)A_1x(\theta,\varphi)\mathrm{d}\theta.$$

As a result we arrive at the equality

$$x(t,\varphi) = K(t)\varphi(0) + \int_{-h}^{0} K(t-\theta-h)A_1x(\theta,\varphi)\mathrm{d}\theta, \quad t \geq 0.$$

As $x(\theta,\varphi) = \varphi(\theta)$ for $\theta \in [-h,0]$, the preceding equality coincides with (2.3). □

2.2 Exponential Stability

We now introduce the stability concept that will be used for system (2.1) in the remainder of this part of the book.

Definition 2.2 ([3]). System (2.1) is said to be exponentially stable if there exist $\gamma \geq 1$ and $\sigma > 0$ such that any solution $x(t,\varphi)$ of the system satisfies the inequality

$$\|x(t,\varphi)\| \leq \gamma \mathrm{e}^{-\sigma t} \|\varphi\|_h, \quad t \geq 0. \tag{2.4}$$

Remark 2.2. It is well known that the exponential stability of system (2.1) is equivalent to the asymptotic stability of the system; see [3].

Definition 2.3. A complex number s_0 is said to be an eigenvalue of system (2.1) if it is a root of the characteristic function,

$$f(s) = \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1\right),$$

of the system. The set

$$\Lambda = \{s \mid f(s) = 0\}$$

is known as the spectrum of the system.

The next statement shows that the property of exponential stability depends on the location of the spectrum of system (2.1).

Theorem 2.2 ([3]). *System (2.1) is exponentially stable if and only if the spectrum of the system lies in the open left half-plane of the complex plane,*

$$Re(s_0) < 0, \quad s_0 \in \Lambda.$$

The following result is a simplified version of the Krasovskii theorem 1.8. It provides sufficient conditions for the exponential stability of system (2.1).

Theorem 2.3. *System (2.1) is exponentially stable if there exists a functional*

$$v : PC([-h,0],R^n) \to R$$

such that the following conditions hold.

1. *For some positive* α_1, α_2

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

2. *For some* $\beta > 0$ *the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) \leq -\beta \|x(t)\|^2, \quad t \geq 0,$$

holds along the solutions of the system.

2.3 Problem Formulation

Motivated by Theorem 2.3 we address in this section the construction of quadratic functionals that satisfy the theorem conditions. Our approach is based on one of the principal ideas of the direct Lyapunov method, which can be formulated in our case as follows. First, select a time derivative, and then compute the functional whose time derivative along the solution of system (2.1) coincides with the selected one. Since the system is linear and time invariant, it seems natural to start with the case where the time derivative is a quadratic form. One technical assumption needed at the beginning, and which will be dropped later, is that system (2.1) is exponentially stable.

Problem 2.1. Let system (2.1) be exponentially stable. Given a quadratic form $w(x) = x^T W x$, find a functional $v_0(\varphi)$, defined on $PC([-h,0],R^n)$, such that along the solutions of the system the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0(x_t) = -x^T(t) W x(t), \quad t \geq 0. \tag{2.5}$$

2.4 Delay-Free Case

Here we would like to explain, using the case of a delay-free system, some principal ideas behind our approach to the solution of Problem 2.1. Let us consider an exponentially stable system of the form

$$\frac{dx}{dt} = Ax. \tag{2.6}$$

By $x(t,x_0)$ we denote the solution of the system with a given initial condition $x(0,x_0) = x_0$. If the initial condition is not important, then we will use the shorthand notation $x(t)$ instead of $x(t,x_0)$.

Given a quadratic form $w(x) = x^T Wx$, we are looking for a Lyapunov function $v(x)$ such that the following equality holds:

$$\left.\frac{d}{dt}v(x)\right|_{(2.6)} = -w(x). \tag{2.7}$$

Control theory textbooks, see [24, 30, 71], teach us that the desired Lyapunov function is also a quadratic form, $v(x) = x^T Vx$. The function satisfies the preceding equality if matrix V is a solution of the classical Lyapunov matrix equation

$$A^T V + VA = -W. \tag{2.8}$$

The evident simplicity of the construction of the Lyapunov function $v(x)$ is achieved due to the valuable information that the function is a quadratic form. Let us assume now that this information is not available. This is exactly the situation that we have now in Problem 2.1. The question is: Is it possible to reveal the form of the Lyapunov function during the construction of $v(x)$? To answer this question, we address ourselves to the Eq. (2.7). First, we observe that if we substitute into (2.7) a solution of system (2.6), then the equation takes the form

$$\frac{d}{dt}v(x(t)) = -w(x(t)).$$

It is evident that the preceding equation defines the function $v(x(t))$ up to an additive constant. To define the constant correctly, we recall that the desired Lyapunov function $v(x)$ should be equal to zero for $x = 0$, $v(0) = 0$. Now we integrate the last equality on the segment $[0,T]$, where $T > 0$,

$$v(x(T,x_0)) - v(x_0) = -\int_0^T x^T(t,x_0)Wx(t,x_0)dt.$$

Since system (2.6) is exponentially stable, $x(T,x_0) \to 0$ and $v(x(T,x_0)) \to 0$, as $T \to \infty$. At the limit we obtain the equality

$$v(x_0) = \int_0^\infty x^T(t,x_0)Wx(t,x_0)\mathrm{d}t. \tag{2.9}$$

The improper integral on the right-hand side of the preceding equality converges due to exponential stability of system (2.6). It is well known that

$$x(t,x_0) = \mathrm{e}^{At}x_0.$$

Replacing $x(t,x_0)$ in (2.9) by this expression we arrive at the equality

$$v(x_0) = x_0^T \left(\int_0^\infty \mathrm{e}^{A^T t} W \mathrm{e}^{At} \mathrm{d}t \right) x_0,$$

which demonstrates that the Lyapunov function is a quadratic form, $v(x) = x^T V x$, with the matrix

$$V = \int_0^\infty \mathrm{e}^{A^T t} W \mathrm{e}^{At} \mathrm{d}t. \tag{2.10}$$

It is a matter of simple calculation to verify that the matrix satisfies Eq. (2.8).

We summarize now the essential elements of the presented construction process. First, the exponential stability assumption allows us to justify formula (2.9). Second, the explicit expression of the solutions of system (2.6) allows us to clarify the form of the desired function. Finally, we see that there is no need to evaluate the improper integral (2.10) to compute the matrix since the matrix Eq. (2.8) serves the same purpose.

It will be shown that this approach to the construction of Lyapunov functions can be extended to the case of time-delay systems in the sense that it is possible to compute a Lyapunov functional that solves Problem 2.1.

2.5 Computation of $v_0(\varphi)$

In this section an important step toward the construction of quadratic functionals that satisfy Theorem 2.3 will be made. That is, we derive an explicit formula for a functional that solves Problem 2.1. We also introduce a Lyapunov matrix that defines the functional.

Equation (2.5) defines the functional $v_0(\varphi)$ up to an additive constant. It follows from the first condition of Theorem 2.3 that this additive constant should be set in such a way that for the trivial initial function $0_h \in PC([-h,0],R^n)$, $v_0(0_h)=0$. Integrating Eq. (2.5) from $t=0$ to $t=T>0$ we obtain

$$v_0(x_T(\varphi)) - v_0(\varphi) = -\int_0^T x(t,\varphi)Wx(t,\varphi)\mathrm{d}t.$$

Since system (2.1) is exponentially stable, $x_T(\varphi) \to 0_h$ as $T\to\infty$, and we arrive at the expression

$$v_0(\varphi) = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t, \quad \varphi \in PC([-h,0],R^n).$$

The exponential stability of system (2.1) implies that the improper integral on the right-hand side of the preceding equality is well defined. If we replace $x(t,\varphi)$ under the integral sign by Cauchy formula (2.3), then

$$\begin{aligned}
v_0(\varphi) &= \int_0^\infty \left[K(t)\varphi(0) + \int_{-h}^0 K(t-h-\theta_1)A_1\varphi(\theta_1)\mathrm{d}\theta_1\right]^T W \\
&\quad \times \left[K(t)\varphi(0) + \int_{-h}^0 K(t-h-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}t \\
&= \varphi^T(0)\left[\int_0^\infty K^T(t)WK(t)\mathrm{d}t\right]\varphi(0) \\
&\quad +2\varphi^T(0)\int_0^\infty K^T(t)W\left[\int_{-h}^0 K(t-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta\right]\mathrm{d}t \\
&\quad +\int_0^\infty \left[\int_{-h}^0 K(t-h-\theta_1)A_1\varphi(\theta_1)\mathrm{d}\theta_1\right]^T W\left[\int_{-h}^0 K(t-h-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}t.
\end{aligned}$$

Here for the first time we encounter the matrix

$$U(\tau) = \int_0^\infty K^T(t)WK(t+\tau)\mathrm{d}t, \tag{2.11}$$

which will play a crucial role in the following study. Since the columns of the fundamental matrix $K(t)$ are solutions of (2.1) with specific initial conditions, it is easy to verify that the matrix admits an upper exponential estimate of the form

$$\|K(t)\| \leq \gamma e^{-\sigma t}, \quad t \geq 0. \tag{2.12}$$

This estimate guarantees that the matrix $U(\tau)$ is well defined for $\tau \in R$.

Lemma 2.1. *Given $\tau_0 \in R$, the improper integral (2.11) converges absolutely and uniformly with respect to $\tau \in [\tau_0, \infty)$.*

Proof. Given $\tau_0 \in R$, it follows directly from (2.12) that

$$\left\|K^T(t)WK(t+\tau)\right\| \leq \gamma^2 \|W\| e^{-\sigma(2t+\tau)}, \quad t \geq 0.$$

Now, let $\tau \in [\tau_0, \infty)$; then the inequality

$$\int_0^\infty \left\|K^T(t)WK(t+\tau)\right\| dt \leq \frac{\gamma^2}{2\sigma} \|W\| e^{-\sigma\tau_0}$$

proves the statement. □

We will demonstrate now that matrix (2.11) allows us to present the functional $v_0(\varphi)$ in a form more suitable for subsequent analysis. To begin with, we observe that the first term of the functional can be written as

$$R_1 = \varphi^T(0)\left[\int_0^\infty K^T(t)WK(t)dt\right]\varphi(0) = \varphi^T(0)U(0)\varphi(0).$$

Lemma 2.1 justifies the following change in the integration order in the second term:

$$\begin{aligned} R_2 &= 2\varphi^T(0)\int_0^\infty K^T(t)W\left[\int_{-h}^0 K(t-h-\theta)A_1\varphi(\theta)d\theta\right]dt \\ &= 2\varphi^T(0)\int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)d\theta. \end{aligned}$$

To present the last term in a similar form, we consider the integral

$$J = \int_0^\infty K^T(t-\tau_1)WK(t-\tau_2)dt,$$

where τ_1 and τ_2 are two nonnegative constants. This integral can be transformed as follows:

$$J = \int_0^\infty K^T(t-\tau_1)WK(t-\tau_2)\mathrm{d}t = \int_{-\tau_1}^\infty K^T(\xi)WK(\xi+\tau_1-\tau_2)\mathrm{d}\xi$$
$$= \int_{-\tau_1}^0 K^T(\xi)WK(\xi+\tau_1-\tau_2)\mathrm{d}\xi + U(\tau_1-\tau_2).$$

Since $K(\xi)=0_{n\times n}$, for $\xi\in[-\tau_1,0)$, the first summand on the right-hand side of the preceding line of equalities disappears, and we obtain

$$J = U(\tau_1-\tau_2).$$

Now, based on Lemma 2.1, we present the last term in the form

$$R_3 = \int_0^\infty \left[\int_{-h}^0 K(t-h-\theta_1)A_1\varphi(\theta_1)\mathrm{d}\theta_1\right]^T W \left[\int_{-h}^0 K(t-h-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}t$$
$$= \int_{-h}^0 \varphi^T(\theta_1)A_1^T\left[\int_{-h}^0\left(\int_0^\infty K^T(t-h-\theta_1)WK(t-h-\theta_2)\mathrm{d}t\right)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$
$$= \int_{-h}^0 \varphi^T(\theta_1)A_1^T\left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1.$$

These transformations show that the quadratic functional $v_0(\varphi)$ can be written as

$$v_0(\varphi) = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta$$
$$+\int_{-h}^0 \varphi^T(\theta_1)A_1^T\left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1. \tag{2.13}$$

Observe that all the terms on the right-hand side of (2.13) depend on the matrix $U(\tau)$. This is the first, but not the last, reason to call this matrix a Lyapunov matrix of system (2.1).

Definition 2.4. Matrix (2.11) is called a Lyapunov matrix of system (2.1) associated with matrix W.

In the following sections the Lyapunov matrices of system (2.1) will be studied in detail. Here we prove only that they are continuous functions of τ.

Lemma 2.2. *Lyapunov matrix (2.11) depends continuously on* $\tau \geq 0$.

Proof. The statement is a direct consequence of Lemma 2.1 and the fact that $K(t)$ is continuous for $t \geq 0$. □

2.6 Lyapunov Matrices: Basic Properties

As was mentioned at the end of the previous section, matrix (2.11) depends continuously on τ. In this section we launch a more detailed analysis of the basic properties of the matrix. Some of the properties will allow us to provide a new definition of the matrix.

Lemma 2.3. *The Lyapunov matrix* $U(\tau)$ *satisfies the dynamic property*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) = U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \geq 0. \tag{2.14}$$

Proof. Let $t \geq 0$ and $\tau > 0$; then

$$\frac{\partial}{\partial\tau}\left[K^T(t)WK(t+\tau)\right] = K^T(t)W\left[K(t+\tau)A_0 + K(t+\tau-h)A_1\right].$$

The exponential stability of system (2.1) implies that

$$\begin{aligned}\left\|\frac{\partial}{\partial\tau}\left[K^T(t)WK(t+\tau)\right]\right\| &\leq \|K(t)\|\,\|W\|\,\|K(t+\tau)\|\,\|A_0\| \\ &\quad + \|K(t)\|\,\|W\|\,\|K(t+\tau-h)\|\,\|A_1\| \\ &\leq \gamma^2 \mathrm{e}^{-\sigma(2t+\tau)}\|W\|\left(\|A_0\| + \mathrm{e}^{\sigma h}\|A_1\|\right) \\ &\leq \gamma^2 \mathrm{e}^{\sigma h}\mathrm{e}^{-2\sigma t}\|W\|\left(\|A_0\| + \|A_1\|\right).\end{aligned}$$

On the one hand, since the integral

$$\int_0^\infty \gamma^2 \mathrm{e}^{\sigma h}\mathrm{e}^{-2\sigma t}\|W\|\left(\|A_0\| + \|A_1\|\right)\mathrm{d}t$$

converges, the integral

$$\int_0^{\infty} \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\mathrm{d}t$$

converges absolutely and uniformly with respect to $\tau \geq 0$, which in turn implies the equality

$$\int_0^{\infty} \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^{\infty} K^T(t)WK(t+\tau)\mathrm{d}t\right) = \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}.$$

On the other hand,

$$\begin{aligned}\int_0^{\infty} \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\mathrm{d}t &= \left(\int_0^{\infty} K^T(t)WK(t+\tau)\mathrm{d}t\right)A_0 \\ &+ \left(\int_0^{\infty} K^T(t)WK(t+\tau-h)\mathrm{d}t\right)A_1 = U(\tau)A_0 + U(\tau-h)A_1,\end{aligned}$$

and we arrive at the dynamic property (2.14). □

Lemma 2.4. *A Lyapunov matrix satisfies the symmetry property*

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0. \tag{2.15}$$

Proof. The matrix

$$\begin{aligned}U(-\tau) &= \int_0^{\infty} K^T(t)WK(t-\tau)\mathrm{d}t = \int_{-\tau}^{\infty} K^T(\xi+\tau)WK(\xi)\mathrm{d}\xi \\ &= \int_{-\tau}^{0} K^T(\xi+\tau)WK(\xi)\mathrm{d}\xi + U^T(\tau).\end{aligned}$$

Since the matrix $K(\xi) = 0_{n\times n}$, $\xi \in [-\tau, 0)$, the integral term on the right-hand side of the preceding line of equalities disappears, and we arrive at (2.15). □

Corollary 2.1. *It follows from (2.15) that matrix $U(0)$ is symmetric, $U^T(0) = U(0)$.*

Corollary 2.2. *Lyapunov matrix (2.11) is infinitely many times differentiable for $\tau \in (0, h)$.*

Indeed, the symmetry property (2.15) and Lemma 2.2 imply that the initial condition for matrix $U(\tau)$ as a solution of Eq. (2.14) is continuous, so the Lyapunov matrix is continuously differentiable. This means, in turn, that the initial condition is continuously differentiable, so the Lyapunov matrix is two times continuously differentiable. It is evident that this process may be continued up to infinity.

Lemma 2.5. *The Lyapunov matrix satisfies the algebraic property*

$$U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) = -W. \tag{2.16}$$

Proof. We first differentiate the product

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[K^T(t)WK(t)\right] &= \left[K(t)A_0 + K(t-h)A_1\right]^T WK(t) \\ &\quad + K^T(t)W\left[K(t)A_0 + K(t-h)A_1\right], \quad t \geq 0,\end{aligned}$$

and then, integrating the preceding equality from $t = 0$ to $t = \infty$, we get

$$\begin{aligned}-W &= \int_0^\infty \left[K(t)A_0 + K(t-h)A_1\right]^T WK(t)\mathrm{d}t \\ &\quad + \int_0^\infty K^T(t)W\left[K(t)A_0 + K(t-h)A_1\right]\mathrm{d}t \\ &= A_0^T U(0) + A_1^T U(h) + U(0)A_0 + U(-h)A_1.\end{aligned}$$

□

The symmetry property indicates that the first derivative of the Lyapunov matrix suffers discontinuity at the point $\tau = 0$.

Lemma 2.6. *Algebraic property (2.16) can be written in the form*

$$U'(+0) - U'(-0) = -W.$$

Here $U'(+0)$ and $U'(-0)$ stand respectively for the right-hand-side and the left-hand-side derivatives of the matrix $U(\tau)$ at $\tau = 0$.

Proof. Observe first that

$$U'(+0) = \lim_{\tau \to +0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = U(0)A_0 + U(-h)A_1.$$

Differentiating the symmetry property we first get

$$\frac{\mathrm{d}U(-\tau)}{\mathrm{d}\tau} = \left[\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right]^T, \quad \tau > 0$$

and then

$$\lim_{\tau\to -0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = -\left[U(0)A_0 + U(-h)A_1\right]^T.$$

Now the statement of the lemma is a simple consequence of the equality

$$\lim_{\tau\to +0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} - \lim_{\tau\to -0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = U(0)A_0 + U(-h)A_1 + \left[U(0)A_0 + U(-h)A_1\right]^T$$

and algebraic property (2.16). □

Despite the fact that functional (2.13) was computed from Eq. (2.5), it seems to be instructive to demonstrate directly that the functional satisfies the equation.

Theorem 2.4. *Functional (2.13), with $U(\tau)$ given by (2.11), satisfies Eq. (2.5).*

Proof. Let $x(t)$ be a solution of system (2.1); then

$$v_0(x_t) = \underbrace{x^T(t)U(0)x(t)}_{R_1(t)} + \underbrace{2x^T(t)\int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)\mathrm{d}\theta}_{R_2(t)}$$
$$+ \underbrace{\int_{-h}^{0} x^T(t+\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1x(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1}_{R_3(t)}.$$

We will differentiate the summands on the right-hand side of the preceding equality one by one.

It is easy to see that for the first term

$$\frac{\mathrm{d}}{\mathrm{d}t}R_1(t) = 2x^T(t)U(0)\left[A_0x(t) + A_1x(t-h)\right]$$
$$= x^T(t)\left[U(0)A_0 + A_0^TU(0)\right]x(t) + 2x^T(t)U(0)A_1x(t-h).$$

By a simple change of the integration variable we present the second term in a form more suitable for subsequent differentiation:

$$R_2(t) = 2x^T(t)\int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) &= 2\left[A_0x(t)+A_1x(t-h)\right]^T \int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi \\
&\quad +2x^T(t)U(-h)A_1x(t)-2x^T(t)U(0)A_1x(t-h) \\
&\quad +2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}U(-h-\xi+t)\right)A_1x(\xi)\mathrm{d}\xi \\
&= 2x^T(t)A_0^T\int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi \\
&\quad +2x^T(t-h)A_1^T\int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi \\
&\quad +2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}U(-h-\xi+t)\right)A_1x(\xi)\mathrm{d}\xi \\
&\quad +x^T(t)\left[U(-h)A_1+A_1^TU(h)\right]x(t) \\
&\quad -2x^T(t)U(0)A_1x(t-h).
\end{aligned}$$

Applying to the last term a similar change of the integration variables we have

$$R_3(t)=\int_{t-h}^{t} x^T(\xi_1)A_1^T\left[\int_{t-h}^{t} U(\xi_1-\xi_2)A_1x(\xi_2)\mathrm{d}\xi_2\right]\mathrm{d}\xi_1.$$

The time derivative of the term is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_3(t) &= x^T(t)A_1^T\int_{t-h}^{t} U(t-\xi)A_1x(\xi)\mathrm{d}\xi \\
&\quad -x^T(t-h)A_1^T\int_{t-h}^{t} U(t-h-\xi)A_1x(\xi)\mathrm{d}\xi
\end{aligned}$$

$$+\left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t)\mathrm{d}\xi\right) A_1 x(t)$$

$$-\left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t + h)\mathrm{d}\xi\right) A_1 x(t-h).$$

Since the term

$$J_1(t) = \left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t)\mathrm{d}\xi\right) A_1 x(t)$$
$$= x^T(t)A_1^T \int_{t-h}^{t} U(t-\xi)A_1 x(\xi)\mathrm{d}\xi$$

and the term

$$J_2(t) = \left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t + h)\mathrm{d}\xi\right) A_1 x(t-h)$$
$$= x^T(t-h)A_1^T \int_{t-h}^{t} U(t-h-\xi)A_1 x(\xi)\mathrm{d}\xi,$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}R_3(t) = 2x^T(t)A_1^T \int_{t-h}^{t} U(t-\xi)A_1 x(\xi)\mathrm{d}\xi$$
$$-2x^T(t-h)A_1^T \int_{t-h}^{t} U(t-h-\xi)A_1 x(\xi)\mathrm{d}\xi.$$

In the computed time derivatives we first collect all terms that include an integral factor. The sum of such terms is

$$S_1(t) = 2x^T(t)\int_{t-h}^{t} \left[A_0^T U(-h-\xi+t) + A_1^T U(t-\xi) + \frac{\partial}{\partial t}U(-h-\xi+t)\right] A_1 x(\xi)\mathrm{d}\xi.$$

Applying symmetry property (2.15) we find that

$$\frac{\partial}{\partial t}U(-h-\xi+t)=\left[\frac{\partial}{\partial t}U(h+\xi-t)\right]^T=-\left[\left(\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\right)\bigg|_{\tau=h+\xi-t}\right]^T.$$

As $\xi\in[t-h,t]$, the variable $\tau=h+\xi-t\geq 0$ and properties (2.14) and (2.15) imply that

$$\frac{\partial}{\partial t}U(-h-\xi+t)=-A_0^TU(-h-\xi+t)-A_1^TU(-\xi+t).$$

The preceding equality means that the sum of the terms with an integral factor disappears.

Now we collect in the computed time derivatives the algebraic terms. The sum of the terms is

$$S_2(t)=x^T(t)\left[U(0)A_0+A_0^TU(0)+U(-h)A_1+A_1^TU(h)\right]x(t).$$

Property (2.16) implies that the preceding quadratic form coincides with $-w(x(t))$; therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t)=-x^T(t)Wx(t),\quad t\geq 0.$$ □

2.7 Lyapunov Matrices: Limit Case

It is interesting to see what happens with Lyapunov matrices of system (2.1) when the delay term disappears. It may occur in two cases.

In the first case matrix A_1 tends to $0_{n\times n}$, and the limit system is of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=A_0x(t).$$

Assume that matrix A_1 tends to $0_{n\times n}$ in such a way that the system remains exponentially stable. It follows from (2.13) that in this case the functional $v_0(x_t)$ tends to the quadratic form $x^T(t)U(0)x(t)$. The symmetry property (2.15) implies that matrix $U(0)$ is symmetric, and the algebraic property (2.16) turns into the classical Lyapunov matrix equation for $U(0)$:

$$U(0)A_0+A_0^TU(0)=-W.$$

In the second case, the delay tends to zero, $h\to +0$, and the new limit system is of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=(A_0+A_1)x(t).$$

Once again, assume that the system remains exponentially stable when $h \to +0$; then functional (2.13) tends to the quadratic form $x^T(t)U(0)x(t)$. Symmetry property (2.15) implies that matrix $U(0)$ is symmetric, and algebraic property (2.16) takes the form of the classical Lyapunov matrix equation

$$U(0)[A_0 + A_1] + [A_0 + A_1]^T U(0) = -W$$

for the new limit system.

This brief analysis provides an additional justification for calling matrix $U(\tau)$ a Lyapunov matrix of system (2.1).

2.8 Lyapunov Matrices: New Definition

Two serious limitations are associated with the definition of Lyapunov matrices by means of improper integral (2.11). The first one is that this definition is applicable to exponentially stable systems only. The second one is that the definition is of little help from a computational point of view. Indeed, it demands a preliminary computation of the fundamental matrix $K(t)$ for $t \in [0,\infty)$, which by itself is a difficult task, and the consequent evaluation of integral (2.11) for different values of τ.

In this section we remove the assumption that system (2.1) is exponentially stable and propose a new definition of the Lyapunov matrices, which will serve for unstable systems as well. But first we prove the following result.

Theorem 2.5. *Let $\widetilde{U}(\tau)$ be a solution of Eq. (2.14) that satisfies properties (2.15) and (2.16). If we define the functional $\widetilde{v}_0(\varphi)$, $\varphi \in PC([-h,0],R^n)$, by formula (2.13), where the matrix $U(\tau)$ is replaced by the matrix $\widetilde{U}(\tau)$, then the functional is such that along the solutions of system (2.1) the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

Proof. A direct inspection shows that the only properties of the matrix $U(\tau)$ that were used in the proof of Theorem 2.4 are those given in (2.14)–(2.16). Since the matrix $\widetilde{U}(\tau)$ satisfies the properties, the functional $\widetilde{v}_0(\varphi)$ satisfies Eq. (2.5) as well. □

Theorem 2.5 justifies the following definition.

Definition 2.5. We say that the matrix $U(\tau)$ is a Lyapunov matrix of system (2.1) associated with a symmetric matrix W if it satisfies properties (2.14)–(2.16).

On the one hand, the new definition makes it possible to overcome the first limitation of the original definition of the Lyapunov matrices – the exponential stability assumption. On the other hand, it poses a new question: Does Definition 2.5

define for the case of exponentially stable system (2.1) the same Lyapunov matrix as that defined by improper integral (2.11)? The following statement provides an affirmative answer to this question.

Theorem 2.6. *Let system (2.1) be exponentially stable. Then matrix (2.11) is the unique solution of Eq. (2.14) that satisfies properties (2.15) and (2.16).*

Proof. Indeed, matrix (2.11) satisfies Eq. (2.14) and properties (2.15) and (2.16) (Lemmas 2.3–2.5). Assume that there are two matrices, $U_j(\tau)$, $j = 1,2$, that satisfy these three properties. We define two functionals of the form (2.13). The first one, $v_0^{(1)}(\varphi)$, with matrix $U(\tau) = U_1(\tau)$, and the other one, $v_0^{(2)}(\varphi)$, with matrix $U(\tau) = U_2(\tau)$. Then, by Theorem 2.5,

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0^{(j)}(x_t) = -x^T(t)Wx(t), \quad j = 1,2.$$

Since the difference $\Delta v(x_t) = v_0^{(2)}(x_t) - v_0^{(1)}(x_t)$ satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(x_t) = 0, \quad t \geq 0,$$

we conclude that the identity

$$\Delta v(x_t(\varphi)) = \Delta v(\varphi), \quad t \geq 0,$$

holds along any solution of system (2.1). The exponential stability of the system implies that $x_t(\varphi) \to 0_h$ as t tends to ∞. This means that $\Delta v(x_t(\varphi)) \to 0$ as t tend to ∞, and we arrive at the conclusion that for any initial function $\varphi \in PC([-h,0],R^n)$ the following equality holds:

$$\begin{aligned} 0 = \Delta v(\varphi) &= \varphi^T(0)\Delta U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} \Delta U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta \\ &+ \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left[\int_{-h}^{0} \Delta U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1. \end{aligned} \tag{2.17}$$

Here matrix $\Delta U(\tau) = U_2(\tau) - U_1(\tau)$.

Let us select a vector $\gamma \in R^n$ and define the initial function

$$\varphi^{(1)}(\theta) = \begin{cases} 0, & \theta \in [-h,0) \\ \gamma, & \theta = 0 \end{cases}.$$

For the initial function equality (2.17) takes the form

$$0 = \Delta v(\varphi^{(1)}) = \gamma^T \Delta U(0)\gamma.$$

The preceding equality holds for an arbitrary vector γ. Because matrix $\Delta U(0)$ is symmetric, we obtain that

$$\Delta U(0) = 0_{n\times n}. \tag{2.18}$$

Given two vectors $\gamma, \mu \in R^n$ and $\tau \in (0,h]$, let us select $\varepsilon > 0$ such that $-\tau+\varepsilon < 0$ and define the new initial function

$$\varphi^{(2)}(\theta) = \begin{cases} \mu, & \theta \in [-\tau, -\tau+\varepsilon), \\ \gamma, & \theta = 0, \\ 0, & \text{for all other points of segment } [-h,0]. \end{cases}$$

It is a matter of direct calculations to demonstrate that

$$\Delta v(\varphi^{(2)}) = 2\int_{-\tau}^{-\tau+\varepsilon} \gamma^T \Delta U(-h-\theta) A_1 \mu \mathrm{d}\theta + \int_{-\tau}^{-\tau+\varepsilon} \left[\int_{-\tau}^{-\tau+\varepsilon} \mu^T A_1^T \Delta U(\theta_1-\theta_2) A_1 \mu \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1.$$

Let ε be sufficiently small; then

$$2\int_{-\tau}^{-\tau+\varepsilon} \gamma^T \Delta U(-h-\theta) A_1 \mu \mathrm{d}\theta = 2\varepsilon\gamma^T \Delta U(\tau-h) A_1 \mu + o(\varepsilon)$$

and

$$\int_{-\tau}^{-\tau+\varepsilon} \left[\int_{-\tau}^{-\tau+\varepsilon} \mu^T A_1^T \Delta U(\theta_1-\theta_2) A_1 \mu \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 = o(\varepsilon).$$

Here the notation $o(\varepsilon)$ stands for a quantity that satisfies the condition

$$\lim_{\varepsilon\to+0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Since the equality $\Delta v(\varphi^{(2)}) = 0$ holds for any sufficiently small $\varepsilon > 0$, we conclude that

$$2\gamma^T \Delta U(\tau-h) A_1 \mu = 0.$$

Because the preceding equality holds for any choice of vectors $\gamma, \mu \in R^n$, we obtain that

$$\Delta U(\tau-h) A_1 = 0_{n\times n}. \tag{2.19}$$

This is true for any $\tau \in (0,h]$. By continuity arguments we obtain that equality (2.19) remains true on the closed segment $[0,h]$.

The matrices $U_j(\tau)$, $j = 1, 2$, satisfy Eq. (2.14), so the matrix $\Delta U(\tau)$ does the same:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0 + \Delta U(\tau - h)A_1, \quad \tau \geq 0.$$

Condition (2.19) implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0, \quad \tau \in [0, h].$$

Because matrix $\Delta U(\tau)$ satisfies (2.18), we immediately obtain the identity

$$\Delta U(\tau) = 0_{n \times n}, \quad \tau \in [0, h],$$

which means that $U_2(\tau) = U_1(\tau)$. This ends the proof. □

2.9 Lyapunov Matrices: Existence and Uniqueness Issues

Definition 2.5 raises the question of when a Lyapunov matrix exists. In other words, we are interested in conditions under which Eq. (2.14) admits a solution that satisfies properties (2.15) and (2.16). Theorem 2.6 provides a partial answer to the question. Here we give a detailed account of the existence and uniqueness of Lyapunov matrices.

First we prove that a Lyapunov matrix $U(\tau)$ provides a solution of a special boundary value problem for an auxiliary system of delay-free linear matrix differential equations. To this end we introduce two auxiliary matrices:

$$Y(\tau) = U(\tau), \quad Z(\tau) = U(\tau - h), \quad \tau \in [0, h]. \tag{2.20}$$

Lemma 2.7. *Let $U(\tau)$ be a Lyapunov matrix associated with a symmetric matrix W; then auxiliary matrices (2.20) satisfy the delay-free system of matrix equations*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau) = Y(\tau)A_0 + Z(\tau)A_1, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) = -A_1^T Y(\tau) - A_0^T Z(\tau) \tag{2.21}$$

and the boundary value conditions

$$Y(0) = Z(h), \qquad A_0^T Y(0) + Y(0)A_0 + A_1^T Y(h) + Z(0)A_1 = -W. \tag{2.22}$$

Proof. The first equation in (2.21) is a direct consequence of Eq. (2.14). To derive the second equation, we observe that $Z(\tau) = U^T(h - \tau)$, $\tau \in [0, h]$, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(h-\tau)\right]^T = -\left[U(h-\tau)A_0 + U(-\tau)A_1\right]^T \\ &= -A_1^T U(\tau) - A_0^T U(\tau - h) = -A_1^T Y(\tau) - A_0^T Z(\tau).\end{aligned}$$

The first boundary value condition follows immediately from (2.20), whereas the second one is the algebraic property (2.16) written in the terms of the auxiliary matrices. □

Now we show that, conversely, any solution of the boundary value problem (2.21) and (2.22) generates a Lyapunov matrix associated with W.

Theorem 2.7. *If a pair $(Y(\tau), Z(\tau))$ satisfies (2.21) and (2.22), then the matrix*

$$U(\tau) = \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right], \tau \in [0,h], \tag{2.23}$$

is a Lyapunov matrix associated with W if we extend it to $[-h,0)$ by setting $U(-\tau) = U^T(\tau)$ for $\tau \in (0,h]$.

Proof. We check that matrix (2.23) satisfies the conditions of Definition 2.5.

Since we define this matrix on $[-h,0)$ by setting $U(-\tau) = U^T(\tau)$, to verify the symmetry property we only need to check that the matrix

$$U(0) = \frac{1}{2}\left[Y(0) + Z^T(h)\right]$$

is symmetric. The first boundary value condition, $Y(0) = Z(h)$, implies that

$$U(0) = \frac{1}{2}\left[Y(0) + Y^T(0)\right],$$

which proves the desired symmetry property.

Now we address the algebraic property. First we observe that the following matrix equalities hold:

$$\begin{aligned} U(0)A_0 + A_0^T U(0) &= \frac{1}{2}\left[Y(0) + Y^T(0)\right] A_0 + \frac{1}{2} A_0^T \left[Y(0) + Y^T(0)\right] \\ &= \frac{1}{2}\left[Y(0)A_0 + A_0^T Y(0)\right] + \frac{1}{2}\left[Y(0)A_0 + A_0^T Y(0)\right]^T \end{aligned}$$

and

$$\begin{aligned} U(-h)A_1 + A_1^T U(h) &= \frac{1}{2}\left[Y(h) + Z^T(0)\right]^T A_1 + \frac{1}{2} A_1^T \left[Y(h) + Z^T(0)\right] \\ &= \frac{1}{2}\left[Z(0)A_1 + A_1^T Y(h)\right] + \frac{1}{2}\left[Z(0)A_1 + A_1^T Y(h)\right]^T. \end{aligned}$$

Therefore,

$$R = U(0)A_0 + A_0^T U(0) + U(-h)A_1 + A_1^T U(h)$$

$$
\begin{aligned}
&= \frac{1}{2}\left[Y(0)A_0 + A_0^T Y(0) + Z(0)A_1 + A_1^T Y(h)\right] \\
&\quad + \frac{1}{2}\left[Y(0)A_0 + A_0^T Y(0) + Z(0)A_1 + A_1^T Y(h)\right]^T.
\end{aligned}
$$

The second boundary value condition in (2.22) implies that

$$
R = -\frac{1}{2}W - \frac{1}{2}W^T = -W.
$$

Finally, we check the dynamic property. The matrix $U(\tau)$ satisfies the equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) &= \frac{1}{2}\frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} + \frac{1}{2}\frac{\mathrm{d}Z^T(h-\tau)}{\mathrm{d}\tau} \\
&= \frac{1}{2}\left[Y(\tau)A_0 + Z(\tau)A_1\right] - \frac{1}{2}\left[-A_1^T Y(h-\tau) - A_0^T Z(h-\tau)\right]^T \\
&= \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right]A_0 + \frac{1}{2}\left[Y(h-\tau) + Z^T(\tau)\right]^T A_1 \\
&= U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \in [0,h].
\end{aligned}
$$

Thus, by Definition 2.5, matrix (2.23) is a Lyapunov matrix associated with W. □

Corollary 2.3. *If the boundary value problem (2.21) and (2.22) admits a unique solution $(Y(\tau), Z(\tau))$, then the matrix*

$$
U(\tau) = Y(\tau), \quad \tau \in [0,h],
$$

is a unique Lyapunov matrix associated with W.

Proof. First we show that if a pair $(Y(\tau), Z(\tau))$ satisfies (2.21) and (2.22), then the pair

$$
\left(\widetilde{Y}(\tau), \widetilde{Z}(\tau)\right) = \left(Z^T(h-\tau), Y^T(h-\tau)\right) \tag{2.24}
$$

also satisfies the conditions. It follows directly from (2.24) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{Y}(\tau) &= -\left[-A_1^T Y(h-\tau) - A_0^T Z(h-\tau)\right]^T = \widetilde{Y}(\tau)A_0 + \widetilde{Z}(\tau)A_1, \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{Z}(\tau) &= -\left[Y(h-\tau)A_0 + Z(h-\tau)A_1\right]^T = -A_1^T\widetilde{Y}(\tau) - A_0^T\widetilde{Z}(\tau).
\end{aligned}
$$

Now, we check that matrices (2.24) satisfy the first boundary value condition in (2.22):

$$
\widetilde{Y}(0) - \widetilde{Z}(h) = [Z(h) - Y(0)]^T = 0_{n\times n}.
$$

And, finally, let us check the second boundary value condition in (2.22):

$$\begin{aligned}\widetilde{R} &= \widetilde{Y}(0)A_0 + A_0^T\widetilde{Y}(0) + A_1^T\widetilde{Y}(h) + \widetilde{Z}(0)A_1\\ &= Z^T(h)A_0 + A_0^T Z^T(h) + A_1^T Z^T(0) + Y^T(h)A_1\\ &= \left[A_0^T Y(0) + Y(0)A_0 + Z(0)A_1 + A_1^T Y(h)\right]^T\\ &= -W.\end{aligned}$$

Because the boundary value problem (2.21) and (2.22) admits a unique solution, we conclude that

$$Y(\tau) = Z^T(h-\tau), \quad \tau \in [0,h],$$

and therefore

$$U(\tau) = \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right] = Y(\tau), \quad \tau \in [0,h],$$

is a Lyapunov matrix associated with W. □

We present now an important condition for system (2.1). For the delay-free case this condition is well known and guarantees that the classical Lyapunov matrix Eq. (2.11) admits a unique solution for any matrix W.

Definition 2.6. We say that system (2.1) satisfies the Lyapunov condition if the spectrum of the system,

$$\Lambda = \left\{ s \mid \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1\right) = 0 \right\},$$

does not contain a point s_0 such that $-s_0$ also belongs to the spectrum, or, put another way, there are no eigenvalues of the system arranged symmetrically with respect to the origin of the complex plane.

Remark 2.3. If system (2.1) satisfies the Lyapunov condition, then it has no eigenvalues on the imaginary axis of the complex plane.

The following statement will play an important role in the proof of Theorem 2.8.

Lemma 2.8. *If system (2.21) admits a solution $(Y(\tau), Z(\tau))$ of the boundary value problem (2.22) with $W = 0_{n\times n}$, then*

$$Y(\tau) = Z(h+\tau), \quad \tau \in R. \tag{2.25}$$

Proof. We verify first that the matrices $Y(\tau)$ and $Z(\tau)$ satisfy the second-order matrix differential equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} = \frac{\mathrm{d}X}{\mathrm{d}\tau}A_0 - A_0^T\frac{\mathrm{d}X}{\mathrm{d}\tau} + A_0^T X A_0 - A_1^T X A_1. \tag{2.26}$$

To this end, we differentiate the first equation of system (2.21):

$$\frac{\mathrm{d}^2 Y(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau}A_0 + \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}A_1.$$

The last term on the right-hand side of the preceding equality can be expressed by means of the second equation of (2.21) as follows:

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}A_1 = -A_1^T Y(\tau)A_1 - A_0^T Z(\tau)A_1.$$

Then

$$\frac{\mathrm{d}^2 Y(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau}A_0 - A_1^T Y(\tau)A_1 - A_0^T Z(\tau)A_1.$$

The first equation of (2.21) allows us to present the last term on the right-hand side of the preceding equality in the form

$$-A_0^T Z(\tau)A_1 = -A_0^T \left[\frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} - Y(\tau)A_0\right].$$

And we arrive at the conclusion that $Y(\tau)$ satisfies Eq. (2.26). Similar manipulations prove that the matrix $Z(\tau)$ is a solution of the equation as well.

Any solution of (2.26) is uniquely determined by the initial conditions, $X(\tau_0)$, $X'(\tau_0)$. For $W = 0_{n\times n}$ the second condition in (2.22) can be transformed as follows:

$$\begin{aligned} 0_{n\times n} &= Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h) \\ &= Y'(0) - Z'(h). \end{aligned}$$

If we add to the preceding equality the first condition from (2.22), $Y(0) = Z(h)$, then the identity (2.25) becomes evident. □

Now everything is ready to present the main result of the section.

Theorem 2.8. *System (2.1) admits a unique Lyapunov matrix associated with a given symmetric matrix W if and only if the system satisfies the Lyapunov condition.*

Proof. Sufficiency: Given a symmetric matrix W, according to Theorem 2.7, we can compute a Lyapunov matrix associated with W if there exists a solution of the boundary value problem (2.21) and (2.22). In what follows, we demonstrate that under the Lyapunov condition the boundary value problem admits a unique solution.

Let system (2.1) satisfy the Lyapunov condition. System (2.21) is linear and time invariant. To define a particular solution of the system, one must know the initial matrices $Y_0 = Y(0)$, $Z_0 = Z(0)$. This means that, in total, the initial matrices have $2n^2$ unknown components. Conditions (2.22) provide a system of $2n^2$ scalar linear algebraic equations in $2n^2$ unknown components of the initial matrices. The algebraic system admits a unique solution if and only if the unique solution of the system with $W = 0_{n\times n}$ is the trivial one. Assume by contradiction that there exists a nontrivial solution, (Y_0, Z_0), of the algebraic system with $W = 0_{n\times n}$. The initial matrices generate a nontrivial solution, $(Y(\tau), Z(\tau))$, $\tau \in [0,h]$, of the boundary value problem (2.21) and (2.22) with $W = 0_{n\times n}$. The nontrivial solution can be presented as a sum of eigenmotions of system (2.21):

$$Y(\tau) = \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} P_\nu(\tau), \quad Z(\tau) = \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} Q_\nu(\tau).$$

Here s_ν, $\nu = 0,1,\ldots,N$, are distinct eigenvalues of system (2.21) and $P_\nu(\tau)$ and $Q_\nu(\tau)$ are polynomials with matrix coefficients. The solution $(Y(\tau), Z(\tau))$ is nontrivial, so at least one of the polynomials $P_\nu(\tau)$, say $P_0(\tau)$, is nontrivial, because otherwise $Y(\tau) \equiv 0_{n\times n}$, and identity (2.25) implies that $Z(\tau) \equiv 0_{n\times n}$. Let polynomial $P_0(\tau)$ be of degree ℓ,

$$P_0(\tau) = \sum_{j=0}^{\ell} \tau^j B_j,$$

where B_j, $j = 0,1,\ldots \ell$, are constant $n\times n$ matrices, and $B_\ell \neq 0_{n\times n}$. It follows from Lemma 2.8 that $Y(\tau) = Z(h+\tau)$, and therefore

$$P_0(\tau) = \mathrm{e}^{s_0 h} Q_0(\tau + h).$$

Hence $Q_0(\tau)$ is also a nontrivial polynomial of degree ℓ,

$$Q_0(\tau) = \sum_{j=0}^{\ell} \tau^j C_j,$$

where $C_\ell = \mathrm{e}^{-s_0 h} B_\ell$.

Taking into account (2.25), we present the first matrix equation in (2.21) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau) = Y(\tau) A_0 + Y(\tau - h) A_1.$$

And we obtain that

$$\begin{aligned} 0_{n\times n} = & \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left[s_\nu P_\nu(\tau) + \frac{\mathrm{d}P_\nu(\tau)}{\mathrm{d}\tau} \right] \\ & - \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left[P_\nu(\tau) A_0 + \mathrm{e}^{-s_\nu h} P_\nu(\tau - h) A_1 \right]. \end{aligned}$$

Because all eigenvalues s_ν, $\nu = 0,1,\ldots,N$, are distinct, the preceding equality implies that for each ν the polynomial identity

$$0_{n\times n} = s_\nu P_\nu(\tau) + \frac{\mathrm{d}P_\nu(\tau)}{\mathrm{d}\tau} - P_\nu(\tau) A_0 - \mathrm{e}^{-s_\nu h} P_\nu(\tau - h) A_1$$

holds. In the polynomial identity for $\nu = 0$ we collect the terms of the highest degree ℓ. The sum of these terms is equal to a zero matrix, so we arrive at the matrix equality

$$B_\ell \left(s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right) = 0_{n\times n}.$$

Because $B_\ell \neq 0_{n\times n}$, the preceding equality holds only if

$$\det \left(s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right) = 0,$$

and we conclude that s_0 is an eigenvalue of the original system (2.1).

The second equation of system (2.21) and the identity $Y(\tau) = Z(\tau + h)$ imply that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Z(\tau) = -A_1^T Z(\tau + h) - A_0^T Z(\tau).$$

The preceding identity generates the new set of polynomial identities

$$0_{n\times n} = s_\nu Q_\nu(\tau) + \frac{\mathrm{d}Q_\nu(\tau)}{\mathrm{d}\tau} + A_1^T Q_\nu(\tau + h) + A_0^T Q_\nu(\tau), \quad \nu = 0, 1, \ldots, N.$$

If in the identity for $\nu = 0$ we collect the terms of the highest degree ℓ, then

$$\left[s_0 I + A_0 + d^{s_0 h} A_1 \right]^T C_\ell = 0_{n\times n}.$$

As $C_\ell \neq 0_{n\times n}$, the preceding equality holds only if

$$\det \left[(-s_0) I - A_0 - d^{-(-s_0)h} A_1 \right] = 0.$$

And we conclude that $-s_0$ is an eigenvalue of system (2.1). This means that system (2.1) does not satisfy the Lyapunov condition. But this contradicts the theorem condition. The contradiction proves that the only solution of the boundary value problem (2.21), (2.22), with $W = 0_{n\times n}$, is the trivial one. Therefore, the boundary value problem (2.21), (2.22) admits a unique solution for any symmetric W, and this solution generates a Lyapunov matrix associated with W (Theorem 2.7).

Necessity: Now let us assume that system (2.1) does not satisfy the Lyapunov condition, i.e., the spectrum of the system contains a point s_0 such that $-s_0$ also belongs to the spectrum. Then there exist nonzero vectors $\gamma, \mu \in C^n$ such that

$$\mu^T \left[s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right] = 0, \quad \left[(-s_0) I - A_0 - \mathrm{e}^{-(-s_0)h} A_1 \right]^T \gamma = 0.$$

We show that in this case there exists a nontrivial solution $(Y(\tau), Z(\tau))$, $\tau \in [0, h]$, of the boundary value problem (2.21), (2.22) with $W = 0_{n\times n}$. To check this, we set $Y(\tau) = \mathrm{e}^{s_0 \tau} \gamma \mu^T$ and $Z(\tau) = \mathrm{e}^{s_0(\tau - h)} \gamma \mu^T$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau) = s_0 \mathrm{e}^{s_0\tau}\gamma\mu^T = \mathrm{e}^{s_0\tau}\gamma\mu^T(A_0 + \mathrm{e}^{-s_0 h}A_1)$$
$$= Y(\tau)A_0 + Z(\tau)A_1,$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) = s_0 \mathrm{e}^{s_0(\tau-h)}\gamma\mu^T = \mathrm{e}^{s_0(\tau-h)}(-A_0^T - \mathrm{e}^{s_0 h}A_1^T)\gamma\mu^T$$
$$= -A_1^T Y(\tau) - A_0^T Z(\tau).$$

It is evident that $Y(\tau) = Z(\tau+h)$, so

$$Y(0) = Z(h), \text{ and } \left.\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)\right|_{\tau=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau)\right|_{\tau=h}.$$

By Theorem 2.7, the nontrivial solution generates the following nontrivial Lyapunov matrix associated with $W = 0_{n\times n}$:

$$U_0(\tau) = \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right] = \frac{1}{2}\left[\mathrm{e}^{s_0\tau}\gamma\mu^T + \mathrm{e}^{-s_0\tau}\mu\gamma^T\right].$$

Assume now that for a given symmetric matrix W there exists a Lyapunov matrix $U(\tau)$. It is evident that the matrix $U(\tau)+U_0(\tau)$ is also a Lyapunov matrix associated with W. This contradicts the theorem condition. The contradiction shows that our assumption that (2.1) does not satisfy the Lyapunov condition is wrong. This ends the proof of the necessity part. □

Corollary 2.4. *The Lyapunov matrix $U(\tau)$ associated with W satisfies the second-order delay-free matrix equation*

$$\frac{\mathrm{d}^2U(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}A_0 - A_0^T\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} + A_0^T U(\tau)A_0 - A_1^T U(\tau)A_0, \qquad \tau \geq 0,$$

and the following boundary value conditions:

$$\left.\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0} = U(0)A_0 + U^T(h)A_1,$$
$$\left.\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0} + \left(\left.\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0}\right)^T = -W.$$

Let us now see what happens when system (2.1) does not satisfy the Lyapunov condition. But first we prove a needed technical result.

Lemma 2.9. *Given two nontrivial vectors $\gamma, \mu \in C^n$, there exists a real symmetric matrix W_0 such that $\gamma^T W_0 \mu \neq 0$.*

Proof. If there exists an index j such that $\gamma_j \mu_j \neq 0$, then the matrix $W_0 = e_j e_j^T$, where e_j denotes the jth unit column vector, satisfies the desired condition $\gamma^T W_0 \mu = \gamma_j \mu_j \neq 0$. If $\gamma_j \mu_j = 0$ for all j, then there exist indices j and k, $j \neq k$, such that $\gamma_j \neq 0$ and $\gamma_k = 0$, while $\mu_j = 0$ and $\mu_k \neq 0$. Hence, setting $W_0 = e_j e_k^T + e_k e_j^T$ we obtain $\gamma^T W_0 \mu = \gamma_j \mu_k \neq 0$. □

Theorem 2.9. *If system (2.1) does not satisfy the Lyapunov condition, then there is a symmetric matrix W for which a Lyapunov matrix associated with W does not exist.*

Proof. Assume by contradiction that for any symmetric matrix W there exists a Lyapunov matrix associated with W. Since system (2.1) does not satisfy the Lyapunov condition, then there exists an eigenvalue s_0 such that $-s_0$ is also an eigenvalue of the system. Let γ and μ be eigenvectors corresponding to these eigenvalues. System (2.1) admits two solutions of the form

$$x^{(1)}(t) = \mathrm{e}^{s_0 t}\gamma, \qquad x^{(2)}(t) = \mathrm{e}^{-s_0 t}\mu.$$

By Lemma 2.9, there exists a symmetric matrix W_0 such that $\gamma^T W_0 \mu \neq 0$. According to our assumption, there is a Lyapunov matrix $U(\tau)$ associated with W_0. Let us define the bilinear functional

$$\begin{aligned} z(\varphi,\psi) &= \varphi^T(0)U(0)\psi(0) + \varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\psi(\theta)\mathrm{d}\theta \\ &+ \left(\int_{-h}^{0} \varphi^T(\theta)A_1^T U(h+\theta)\mathrm{d}\theta\right)\psi(0) \\ &+ \int_{-h}^{0} \varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1\psi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1. \end{aligned}$$

Here it is assumed that $\varphi, \psi \in PC([-h,0],R^n)$. Given two solutions of (2.1), $x(t,\varphi)$ and $x(t,\psi)$, one can verify by direct calculation that

$$\frac{\mathrm{d}z(x_t(\varphi),x_t(\psi))}{\mathrm{d}t} = -x^T(t,\varphi)W_0 x(t,\psi), \quad t \geq 0.$$

On the one hand, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t} z(x_t^{(1)}, x_t^{(2)}) = -\left[x^{(1)}(t)\right]^T W_0 x^{(2)}(t) = -\gamma^T W_0 \mu \neq 0. \tag{2.27}$$

On the other hand, the direct substitution of the solutions into the bilinear functional yields

$$z(x_t^{(1)}, x_t^{(2)}) = \gamma^T \left[U(0) + \int_{-h}^{0} U(-h-\theta)A_1 \mathrm{e}^{s_0\theta} \mathrm{d}\theta + \int_{-h}^{0} A_1^T U(h+\theta)\mathrm{e}^{-s_0\theta}\mathrm{d}\theta \right.$$
$$\left. + \int_{-h}^{0} \mathrm{e}^{s_0\theta_1} A_1^T \left(\int_{-h}^{0} \mathrm{e}^{-s_0\theta_2} U(\theta_1 - \theta_2)\mathrm{d}\theta_2 \right) A_1 \mathrm{d}\theta_1 \right] \mu.$$

Observe that the matrix in the square brackets on the right-hand side of the preceding equality does not depend on t and

$$\frac{\mathrm{d}}{\mathrm{d}t} z(x_t^{(1)}, x_t^{(2)}) = 0, \quad t \geq 0.$$

But this contradicts inequality (2.27). Hence, our assumption is not true, and for symmetric matrix W_0 the associated Lyapunov matrix does not exist. □

2.10 Lyapunov Matrices: Computational Issue

It is evident that the availability of constructive procedures for the computation of the Lyapunov matrices is crucial for a successful application of the quadratic functionals to the analysis of time-delay systems. It was shown in the previous section that the computation of Lyapunov matrices can be reduced to the construction of a solution of a special boundary value problem for a system of delay-free linear matrix differential equations (see Theorem 2.7). In Theorem 2.8 it was shown that the Lyapunov condition guarantees that the boundary value problem admits a unique solution.

With the help of the Kronecker product of matrices [13, 15, 25] the matrix boundary value problem (2.21), (2.22) can be written in vector form, which simplifies the computation of the solution of the problem. To this end, we define the vectorization operation

$$\text{vec}\,(Q) = q,$$

where $q \in R^{n^2}$ is obtained from $Q \in R^{n\times n}$ by stacking up its columns. The operation satisfies the equality

$$\text{vec}\,(AQB) = (A \otimes B)q,$$

where the matrix

$$A \otimes B = \begin{pmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{21}A & & b_{nn}A \end{pmatrix}$$

is known as the Kronecker product of matrices A and B.

System (2.28) takes the vector form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = L\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix}, \qquad L = \begin{pmatrix} I\otimes A_0 & I\otimes A_1 \\ -A_1^T\otimes I & -A_0^T\otimes I \end{pmatrix}. \tag{2.28}$$

Here $y(\tau) = \mathrm{vec}(Y(\tau))$ and $z(\tau) = \mathrm{vec}(Z(\tau))$. Boundary value conditions (2.22) take the form

$$M\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} + N\begin{pmatrix} y(h) \\ z(h) \end{pmatrix} = -\begin{pmatrix} 0 \\ w \end{pmatrix}, \tag{2.29}$$

where $w = \mathrm{vec}\,(W)$ and

$$M = \begin{pmatrix} I\otimes I & 0_{n\times n}\otimes 0_{n\times n} \\ A_0^T\otimes I + I\otimes A_0 & I\otimes A_1 \end{pmatrix},\; N = \begin{pmatrix} 0_{n\times n}\otimes 0_{n\times n} & -I\otimes I \\ A_1^T\otimes I & 0_{n\times n}\otimes 0_{n\times n} \end{pmatrix}. \tag{2.30}$$

It follows from system (2.28) that

$$\begin{pmatrix} y(h) \\ z(h) \end{pmatrix} = \mathrm{e}^{Lh}\begin{pmatrix} y(0) \\ z(0) \end{pmatrix}.$$

Substituting the preceding equality into boundary value condition (2.29) we obtain the algebraic system for the initial vectors

$$\left[M + N\mathrm{e}^{Lh}\right]\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = -\begin{pmatrix} 0 \\ w \end{pmatrix}. \tag{2.31}$$

Assume that the preceding system admits a solution; then this solution generates the corresponding solution of system (2.28),

$$\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = \mathrm{e}^{L\tau}\begin{pmatrix} y(0) \\ z(0) \end{pmatrix},$$

and a solution $(Y(\tau), Z(\tau))$ of the boundary value problem (2.21), (2.22). By Theorem 2.7 we obtain a Lyapunov matrix, associated with W; see (2.23).

We conclude this section with a criterion that system (2.1) satisfies the Lyapunov condition.

Theorem 2.10. *System (2.1) satisfies the Lyapunov condition if and only if the following condition holds:*

$$\det\left(M + N\mathrm{e}^{Lh}\right) \neq 0.$$

Proof. Necessity: System (2.1) satisfies the Lyapunov condition. It has been shown in the proof of the necessity part of Theorem 2.8 that under this condition the only solution of boundary value problem (2.21), (2.22), with $W = 0_{n\times n}$, is the trivial one.

Therefore, the only solution of system (2.31), with $w=0$, is the trivial one. This implies that the matrix $\left(M+N\mathrm{e}^{Lh}\right)$ is nonsingular.

Sufficiency: Because the matrix $\left(M+N\mathrm{e}^{Lh}\right)$ is nonsingular, for any given w system (2.31) admits a unique solution. Therefore, by Corollary 2.3, for any symmetric matrix W system (2.1) admits a unique Lyapunov matrix $U(\tau)$. According to Theorem 2.8, this implies that system (2.1) satisfies the Lyapunov condition. □

2.11 Complete Type Functionals

One of the conditions of Theorem 2.3 states that the functional $v(\varphi)$ should admit a quadratic lower bound of the form

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n),$$

where $\alpha_1 > 0$. Surprisingly enough, no such bound has been found for functional (2.13). An attempt, undertaken in [26], resulted only in a local cubic lower bound for the functional. The following example confirms that no such quadratic lower bound for the functional exists.

Example 2.1 (A.P. Zhabko). The scalar equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t-1), \quad t \geq 0,$$

is exponentially stable. This means that there exist $\gamma \geq 1$ and $\sigma > 0$ such that the inequality

$$|x(t,\varphi)| \leq \gamma \mathrm{e}^{-\sigma t} \|\varphi\|_1, \quad t \geq 0,$$

holds along any solution of the equation.

For a given $\varepsilon \in (0,1)$ we define the initial function

$$\widetilde{\varphi}(\theta) = \begin{cases} \varepsilon, & \theta \in [-1,-1+\varepsilon) \\ 0, & \theta \in [-1+\varepsilon,0) \\ \varepsilon^2, & \theta = 0. \end{cases}$$

It is clear that $\|\widetilde{\varphi}\|_1 = \sup_{\theta\in[-1,0]} |\widetilde{\varphi}(\theta)| = \varepsilon$. The corresponding solution, $x(t,\widetilde{\varphi})$, evaluated by the step-by-step method is of the following form:

For $t \in [0,1]$

$$x(t,\widetilde{\varphi}) = \begin{cases} \varepsilon(\varepsilon - t), & t \in [0,\varepsilon], \\ 0, & t \in (\varepsilon,1]; \end{cases}$$

For $t \in [1,2]$

$$x(t,\widetilde{\varphi}) = \begin{cases} -\varepsilon^2 (t-1) + \frac{1}{2}\varepsilon (t-1)^2, & t \in [1, 1+\varepsilon], \\ -\frac{1}{2}\varepsilon^3, & t \in (1+\varepsilon, 2]. \end{cases}$$

Since $|x(t,\widetilde{\varphi})| \leq \frac{1}{2}\varepsilon^3$ for $t \in [1,2]$, the exponential stability of the equation implies that the inequality

$$|x(t,\widetilde{\varphi})| \leq \gamma \frac{1}{2}\varepsilon^3 \mathrm{e}^{-\sigma(t-2)}$$

holds for $t \geq 2$. We now estimate the value $v_0(\widetilde{\varphi})$:

$$\begin{aligned} v_0(\widetilde{\varphi}) &= \int_0^\infty x^2(t,\widetilde{\varphi})\mathrm{d}t = \int_0^1 x^2(t,\widetilde{\varphi})\mathrm{d}t + \int_1^2 x^2(t,\widetilde{\varphi})\mathrm{d}t + \int_2^\infty x^2(t,\widetilde{\varphi})\mathrm{d}t \\ &\leq \frac{1}{3}\varepsilon^5 + \frac{1}{4}\left(1 + \frac{\gamma^2}{2\sigma}\right)\varepsilon^6. \end{aligned}$$

This demonstrates that the functional $v_0(\varphi)$ does not allow a quadratic lower bound of the form $\alpha_1 |\varphi(0)|^2 \leq v_0(\varphi)$ with $\alpha_1 > 0$; otherwise the inequality

$$\alpha_1 \varepsilon^2 \leq \frac{1}{3}\varepsilon^5 + \frac{1}{4}\left(1 + \frac{\gamma^2}{2\sigma}\right)\varepsilon^6$$

should hold for any $\varepsilon \in (0,1)$.

The preceding example shows that to obtain a functional satisfying the conditions of Theorem 2.3, we need a certain modification of functional (2.13). We are now ready to present this modification.

Theorem 2.11. *Given three symmetric matrices W_j, $j = 0,1,2$, let us define the functional*

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h) + \int_{-h}^{0} \varphi^T(\theta)W_2\varphi(\theta)\mathrm{d}\theta. \tag{2.32}$$

If there exists a Lyapunov matrix $U(\tau)$ associated with the matrix $W = W_0 + W_1 + hW_2$ and $v_0(\varphi)$ is functional (2.13) with this Lyapunov matrix, then the time derivative of the modified functional

$$v(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta \tag{2.33}$$

along the solutions of system (2.1) is such that the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t) = -w(x_t), \quad t \geq 0.$$

Proof. Indeed, the time derivative of the first term, $v_0(x_t)$, is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)\left[W_0 + W_1 + hW_2\right]x(t).$$

The time derivative of the second term

$$\begin{aligned} R(t) &= \int_{-h}^{0} x^T(t+\theta)\left[W_1 + (h+\theta)W_2\right]x(t+\theta)\mathrm{d}\theta \\ &= \int_{t-h}^{t} x^T(s)\left[W_1 + (h+s-t)W_2\right]x(s)\mathrm{d}s \end{aligned}$$

is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}R(t) = x^T(t)\left[W_1 + hW_2\right]x(t) - x^T(t-h)W_1x(t-h) - \int_{t-h}^{t} x^T(s)W_2x(s)\mathrm{d}s.$$

The sum of the time derivatives coincides with $-w(x_t)$. □

Definition 2.7. We say that functional (2.33) is of the complete type if matrices W_j, $j = 0,1,2$, are positive definite.

Lemma 2.10. *Let system (2.1) be exponentially stable. Given positive-definite matrices W_j, $j = 0,1,2$, there exists $\alpha_1 > 0$ such that the complete type functional (2.33) admits the following quadratic lower bound:*

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \tag{2.34}$$

Proof. Consider the modified functional

$$\widetilde{v}(\varphi) = v(\varphi) - \alpha \|\varphi(0)\|^2.$$

Here α is a real parameter. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t),$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\alpha x^T(t)\left[A_0x(t) + A_1x(t-h)\right]\\ &\geq (x^T(t), x^T(t-h))W(\alpha)\begin{pmatrix} x(t)\\ x(t-h)\end{pmatrix}.\end{aligned}$$

The matrix

$$W(\alpha) = \begin{pmatrix} W_0 & 0_{n\times n}\\ 0_{n\times n} & W_1\end{pmatrix} + \alpha\begin{pmatrix} A_0 + A_0^T & A_1\\ A_1^T & 0_{n\times n}\end{pmatrix}.$$

Since the block diagonal matrix on the right-hand side of the preceding equality is positive definite, there exists $\alpha = \alpha_1 > 0$ such that the matrix $W(\alpha_1)$ is positive definite, too. This means that for $\alpha = \alpha_1$ the functional $\widetilde{w}(x_t) \geq 0$. The exponential stability of system (2.1) makes it possible to present the modified functional as follows:

$$\widetilde{v}(\varphi) = \int_0^{\infty} \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0.$$

The last inequality proves that (2.34) holds for $\alpha_1 > 0$. □

Lemma 2.11. *Let system (2.1) satisfy the Lyapunov condition (Definition 2.6). Given symmetric matrices W_j, $j = 0,1,2$, for some positive α_2 functional (2.33) satisfies the inequality*

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

Proof. To prove the inequality, we introduce the following notations:

$$\nu = \max_{\theta\in[0,h]} \|U(\theta)\|, \quad a = \|A_1\|.$$

Now we estimate the terms of functional (2.33). It is evident that

$$R_1 = \varphi^T(0)U(0)\varphi(0) \leq \nu\|\varphi(0)\|^2 \leq \nu\|\varphi\|_h^2$$

and

$$R_2 = 2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta \leq 2\nu ah\|\varphi\|_h^2.$$

For the next term we have

$$\begin{aligned}R_3 &= \int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1\\ &\leq \nu a^2h^2\|\varphi\|_h^2.\end{aligned}$$

Finally, we estimate the additional term as follows:

$$R_4 = \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta$$
$$\leq h\left(\|W_1\| + h\|W_2\|\right)\|\varphi\|_h^2.$$

Collecting the estimations we conclude that

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2,$$

where

$$\alpha_2 = \nu\left(1+ah\right)^2 + h\left(\|W_1\| + h\|W_2\|\right).$$

□

We return now to Theorem 2.3 and show that its conditions are necessary for the exponential stability of system (2.1).

Theorem 2.12. *System (2.1) is exponentially stable if and only if there exists a functional $v: PC([-h,0],R^n) \to R$ such that the following conditions are satisfied.*

1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2$, *for some positive α_1, α_2.*
2. *For some $\beta > 0$ the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) \leq -\beta \|x(t)\|^2, \quad t \geq 0,$$

holds along the solutions of the system.

Proof. Sufficiency follows from Theorem 2.3.

Necessity is a direct consequence of Lemmas 2.10 and 2.11. □

2.12 Applications

In this section we present some applications of Lyapunov matrices and quadratic functionals.

2.12.1 Quadratic Performance Index

We consider a control system of the form

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t).$$

Given a control law

$$\widetilde{u}(t) = Mx(t-h), \quad t \geq 0, \tag{2.35}$$

a closed-loop system is of the form

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-h), \quad t \geq 0, \tag{2.36}$$

where $A_0 = A$ and $A_1 = BM$.

Assume that the closed-loop system is exponentially stable, and define the value of the quadratic performance index

$$J(\widetilde{u}) = \int_0^{\infty} \left[y^T(t) P y(t) + u^T(t) Q u(t) \right] dt. \tag{2.37}$$

Here P and Q are given symmetric matrices of the appropriate dimensions. The value of the index can be presented in the form

$$J(\widetilde{u}) = \int_0^{\infty} \left[x^T(t,\varphi) W_0 x(t,\varphi) + x^T(t-h,\varphi) W_1 x(t-h,\varphi) \right] dt,$$

where $\varphi \in PC([-h,0],R^n)$ is an initial function of the solution $x(t,\varphi)$ of the closed-loop system (2.36) and the matrices $W_0 = C^T PC$ and $W_1 = M^T QM$.

Theorem 2.13. *The value of the performance index (2.37) for the stabilizing control law (2.35) is equal to*

$$J(\widetilde{u}) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta) W_1 \varphi(\theta) d\theta,$$

where $v_0(\varphi)$ is functional (2.13) computed for the Lyapunov matrix $U(\tau)$ associated with the matrix $W = W_0 + W_1 = C^T PC + M^T QM$.

2.12.2 Exponential Estimates

In this section we apply the complete type functionals, defined in the previous section, to derive an exponential estimate for the solutions of system (2.1). We begin with the following statement.

Lemma 2.12. *Let system (2.1) be exponentially stable. Given positive-definite matrices* $W_j,\ j = 0,1,2$, *for the complete type functional (2.33), there exist positive constants* β_ℓ, $\ell = 1,2$, *such that*

$$\beta_1 \|\varphi(0)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \tag{2.38}$$

Proof. To prove inequality (2.38), we consider the functional

$$\widetilde{v}(\varphi) = v(\varphi) - \beta_1 \|\varphi(0)\|^2 - \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where β_1 and β_2 are real parameters. Along the solutions of system (2.1) the functional is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t), \quad t \geq 0.$$

Here

$$\begin{aligned}
\widetilde{w}(x_t) &= w(x_t) + 2\beta_1 x^T(t)\left[A_0 x(t) + A_1 x(t-h)\right] + \beta_2 \left[\|x(t)\|^2 - \|x(t-h)\|^2\right] \\
&\geq \left[x^T(t), x^T(t-h)\right] Q(\beta_1,\beta_2) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.
\end{aligned}$$

The matrix

$$Q(\beta_1,\beta_2) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \beta_1 \begin{pmatrix} A_0 + A_0^T & A_1 \\ A_1^T & 0_{n\times n} \end{pmatrix} + \beta_2 \begin{pmatrix} I & 0_{n\times n} \\ 0_{n\times n} & -I \end{pmatrix}.$$

Since the matrices W_0 and W_1 are positive definite, there exist positive constants β_1, β_2 for which the matrix $Q(\beta_1,\beta_2)$ is positive definite. For such a choice of the parameters the following inequality holds along the solutions of system (2.1):

$$\widetilde{w}(x_t) \geq 0, \quad t \geq 0.$$

The preceding inequality implies that

$$\widetilde{v}(\varphi) = \int_{0}^{\infty} \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0,$$

whence (2.38) follows immediately. □

Lemma 2.13. *Let system (2.1) satisfy the Lyapunov condition (Definition 2.6). Given symmetric matrices* $W_j,\ j = 0,1,2$, *for functional (2.33), there exist positive* *constants* δ_ℓ, $\ell = 1, 2$, *such that*

$$v(\varphi) \leq \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n). \tag{2.39}$$

Proof. We will use the notations introduced in the proof of Lemma 2.11. It is evident that the first two terms of the complete type functional (2.33) admit the upper bounds

$$R_1 = \varphi^T(0)U(0)\varphi(0) \leq \nu \|\varphi(0)\|^2$$

and

$$R_2 = 2\varphi^T(0) \int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta$$

$$\leq \nu a h \|\varphi(0)\|^2 + \nu a \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

For the next term we have

$$R_3 = \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

$$\leq \nu a^2 \left[\int_{-h}^{0} \|\varphi(\theta)\| \mathrm{d}\theta \right]^2 \leq \nu a^2 h \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

Finally, we estimate the additional term

$$R_4 = \int_{-h}^{0} \varphi^T(\theta) \left[W_1 + (h+\theta)W_2\right] \varphi(\theta)\mathrm{d}\theta$$

$$\leq (\|W_1\| + h\|W_2\|) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

Collecting the estimations we conclude that

$$v(\varphi) \leq \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where

$$\delta_1 = \nu(1+ah), \quad \delta_2 = a\delta_1 + (\|W_1\| + h\|W_2\|).$$

□

We show how an exponential estimate for the solutions of system (2.1) can be derived with the use of complete type functionals.

Theorem 2.14. *System (2.1) is exponentially stable if and only if it admits a complete type functional $v(\varphi)$ such that for some $\alpha_1 > 0$ and $\alpha_2 > 0$ the following condition is satisfied:*

$$\alpha_1 \|\varphi(0)\|^2 \le v(\varphi) \le \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

Proof. Sufficiency: Let $v(\varphi)$ be a complete type functional that satisfies the theorem conditions. There exist positive-definite matrices $W_j,\ j=0,1,2$ such that the functional satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t) = -w(x_t), \quad t \ge 0,$$

where

$$w(x_t) = x^T(t)W_0x(t) + x^T(t-h)W_1x(t-h) + \int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)\mathrm{d}\theta.$$

First we show that there exists $\sigma > 0$ for which the inequality

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t} + 2\sigma v(x_t) \le 0, \quad t \ge 0, \tag{2.40}$$

holds. Indeed, on the one hand, according to Lemma 2.13, we can find positive $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$v(\varphi) \le \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n).$$

On the other hand, it is evident that

$$w(\varphi) \ge \lambda_{\min}(W_0)\|\varphi(0)\|^2 + \lambda_{\min}(W_2)\int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n),$$

where $\lambda_{\min}(W)$ stands for the minimal eigenvalue of a symmetric matrix W. We take $\sigma > 0$, which satisfies the inequalities

$$2\sigma\delta_1 \leq \lambda_{\min}(W_0), \quad 2\sigma\delta_2 \leq \lambda_{\min}(W_2).$$

It is evident that such σ satisfies (2.40).

Now, inequality (2.40) implies that

$$v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

Then the theorem condition makes it possible to derive the inequalities

$$\alpha_1 \left\|x(t,\varphi)\right\|^2 \leq v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t} \leq \alpha_2 \left\|\varphi\right\|_h^2 \mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

And we arrive at the desired exponential estimate

$$\|x(t,\varphi)\| \leq \gamma \|\varphi\|_h \mathrm{e}^{-\sigma t}, \quad t \geq 0,$$

where

$$\gamma = \sqrt{\frac{\alpha_2}{\alpha_1}}.$$

Necessity: This part of the proof follows from Theorem 2.11 and Lemmas 2.10 and 2.11. □

Remark 2.4. It is worth mentioning that the exponential estimate obtained by Theorem 2.14 depends on the choice of positive-definite matrices W_j, $j = 0,1,2$. These matrices may serve as free parameters for optimization of the estimate. A special choice of matrices W_j, $j = 0,1,2$, may result in a tighter exponential estimate for the solutions of system (2.1). We do not try here to optimize the estimate.

We can obtain an exponential estimate for the solutions of a time-delay system even if it is not exponentially stable. Indeed, assume that the spectrum of system (2.1),

$$\Lambda = \left\{ s \;\middle|\; \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1\right) = 0 \right\},$$

lies in the half-plane

$$\Gamma = \{ s \;|Re(s) < \Delta \}.$$

Then the spectrum of the modified system,

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = (A_0 - \Delta I) y(t) + \mathrm{e}^{-\Delta h} A_1 y(t-h), \quad t \geq 0,$$

is as follows:

$$\begin{aligned}\widetilde{\Lambda} &= \left\{ s \;\middle|\; \det\left[(s+\Delta)I - A_0 - \mathrm{e}^{-(s+\Delta)h}A_1\right] = 0 \right\} \\ &= \{ s - \Delta \mid s \in \Lambda \}.\end{aligned}$$

This implies that the modified system is exponentially stable. Observe that the solutions of these systems satisfy the identity

$$y(t,\widetilde{\varphi}) = \mathrm{e}^{-\Delta t}x(t,\varphi), \quad t \geq -h, \tag{2.41}$$

where $\widetilde{\varphi}(\theta) = \mathrm{e}^{-\Delta\theta}\varphi(\theta)$, $\theta \in [-h,0]$. Since the modified system is exponentially stable, we can apply Theorem 2.14 to the system and compute $\widetilde{\gamma} \geq 1$ and $\widetilde{\sigma} > 0$ such that

$$\|y(t,\widetilde{\varphi})\| \leq \widetilde{\gamma}\|\widetilde{\varphi}\|_h \mathrm{e}^{-\widetilde{\sigma}t}, \quad t \geq 0.$$

It follows from identity (2.41) that the solutions of system (2.1) satisfy the inequality

$$\|x(t,\varphi)\| \leq \gamma\|\varphi\|_h \mathrm{e}^{\sigma t}, \quad t \geq 0,$$

where $\gamma = \mathrm{e}^{|\Delta|h}\widetilde{\gamma}$, and $\sigma = \widetilde{\sigma} + \Delta$. We may apply this procedure for $\Delta = \Delta_0 + \varepsilon$, where

$$\Delta_0 = \max_{s\in\Lambda} Re(s)$$

and ε is a given positive number.

2.12.3 Critical Values of Delay

In this section an interesting connection between the spectrum of the original system (2.1) and that of the auxiliary system (2.21) will be established. But first we prove the following theorem.

Theorem 2.15. *Let λ_0 be an eigenvalue of system (2.21). Then $-\lambda_0$ is also an eigenvalue of the system.*

Proof. Because λ_0 is an eigenvalue of system (2.21), there exists a nontrivial pair (Y_0, Z_0) of $n \times n$ matrices such that

$$\lambda_0 Y_0 = Y_0 A_0 + Z_0 A_1, \quad \lambda_0 Z_0 = -A_1^T Y_0 - A_0^T Z_0.$$

Applying the transposition to the preceding matrix equalities we obtain that the pair

$$(Y_1, Z_1) = \left(Z_0^T, Y_0^T\right)$$

satisfies the following matrix equalities:

$$-\lambda_0 Y_1 = Y_1 A_0 + Z_1 A_1, \quad -\lambda_0 Z_1 = -A_1^T Y_1 - A_0^T Z_1.$$

This means that $-\lambda_0$ is also an eigenvalue of system (2.21). □

The following theorem provides a connection between the spectrum of time-delay system (2.1) and that of delay-free system (2.21). This connection may be effectively used for the computation of critical delay values of system (2.1), i.e., the values for which system (2.1) admits an eigenvalue on the imaginary axis of the complex plane.

Theorem 2.16. *If s_0 is an eigenvalue of time-delay system (2.1) such that $-s_0$ is also an eigenvalue of the system, then s_0 and $-s_0$ belong to the spectrum of delay-free system (2.21).*

Proof. Since points s_0 and $-s_0$ belong to the spectrum of time-delay system (2.1), there exist two nonzero vectors $\gamma, \mu \in \mathbb{C}^n$ such that

$$\mu^T \left[s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right] = 0,$$

$$\left[(-s_0) I - A_0 - \mathrm{e}^{-(-s_0) h} A_1 \right]^T \gamma = 0.$$

Now, premultiplying the first equality by γ and postmultiplying the second one by $\mathrm{e}^{-s_0 h} \mu^T$, we obtain

$$\gamma \mu^T \left[s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right] = 0_{n \times n},$$

$$\left[(-s_0) I - A_0 - \mathrm{e}^{-(-s_0) h} A_1 \right]^T \mathrm{e}^{-s_0 h} \gamma \mu^T = 0_{n \times n}.$$

If we set $Y_0 = \gamma \mu^T$ and $Z_0 = \mathrm{e}^{-s_0 h} \gamma \mu^T$, then the preceding equalities take the form

$$s_0 Y_0 = Y_0 A_0 + Z_0 A_1, \quad \text{and} \quad s_0 Z_0 = -A_1^T Y_0 - A_0^T Z_0.$$

This means that s_0 belongs to the spectrum of delay-free system (2.21). By Theorem 2.15, $-s_0$ belongs to the spectrum as well. □

Remark 2.5. The spectrum Λ of delay system (2.1) depends on the delay value h, whereas the spectrum of delay-free system (2.21) does not depend on h. In particular, this means that system (2.1) remains exponentially stable (unstable) for all values of delay $h \geq 0$ if the spectrum of system (2.21) has no common points with the imaginary axis of the complex plane. On the other hand, the common points represent possible crossing points of the imaginary axis through which eigenvalues of system (2.1) may migrate from one half-plane to the other as the system delay h varies.

2.12.4 Robustness Bounds

It is well known that Lyapunov functions for delay-free systems are effectively used for estimating the robustness bounds for perturbed systems. The main contribution of this section consists in the demonstration that complete type functionals may also provide reasonable robustness bounds for time-delay systems.

Consider a perturbed system of the form

$$\frac{dy(t)}{dt} = (A_0 + \Delta_0)\,y(t) + (A_1 + \Delta_1)\,y(t-h), \quad t \geq 0. \tag{2.42}$$

Here matrices Δ_0 and Δ_1 are unknown but such that

$$\|\Delta_k\| \leq \rho_k,\; k = 0, 1. \tag{2.43}$$

Let system (2.1) be exponentially stable. We would like to find some conditions on ρ_0 and ρ_1 under which system (2.42) remains stable for all Δ_0 and Δ_1 satisfying (2.43). To this end, we will use a complete type functional $v(\varphi)$ defined by formula (2.33).

We compute the time derivative of the functional along the solutions of perturbed system (2.42).

Lemma 2.14. *The time derivative of functional (2.33) along the solutions of perturbed system (2.42) is of the form*

$$\frac{d}{dt}v(y_t) = -w(y_t) + 2\left[\Delta_0 y(t) + \Delta_1 y(t-h)\right]^T l(y_t), \quad t \geq 0,$$

where

$$l(y_t) = U(0)y(t) + \int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)d\theta.$$

Proof. Recall that $v(y_t)$ is written as follows:

$$\begin{aligned} v_0(y_t) &= y^T(t)U(0)y(t) + 2y^T(t)\int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)d\theta \\ &\quad + \int_{-h}^{0} y^T(t+\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1 y(t+\theta_2)d\theta_2\right] d\theta_1 \\ &\quad + \int_{-h}^{0} y^T(t+\theta)\left[W_1 + (h+\theta)W_2\right] y(t+\theta)d\theta. \end{aligned}$$

The time derivative of the first term,

$$R_1(t) = y^T(t)U(0)y(t),$$

is

$$\frac{\mathrm{d}R_1(t)}{\mathrm{d}t} = 2y^T(t)U(0)\left[(A_0+\Delta_0)y(t)+(A_1+\Delta_1)y(t-h)\right].$$

For the next term,

$$R_2(t) = 2y^T(t)\int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta,$$

we have

$$\begin{aligned}\frac{\mathrm{d}R_2(t)}{\mathrm{d}t} &= 2\left[(A_0+\Delta_0)y(t)+(A_1+\Delta_1)y(t-h)\right]^T\int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta \\ &+2y^T(t)U(-h)A_1y(t)-2y^T(t)U(0)A_1y(t-h) \\ &-2y^T(t)\int_{-h}^{0}\left[U'(h+\theta)\right]^T A_1 y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

The time derivative of the double integral

$$R_3(t) = \int_{-h}^{0} y^T(t+\theta_1)A_1^T\left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1 y(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$

is of the form

$$\begin{aligned}\frac{\mathrm{d}R_3(t)}{\mathrm{d}t} &= 2y^T(t)\int_{-h}^{0}\left[U(\theta)A_1\right]^T A_1 y(t+\theta)\mathrm{d}\theta \\ &\quad -2y^T(t-h)A_1^T\int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

And, finally, the time derivative of the last term,

$$R_4 = \int_{-h}^{0} y^T(t+\theta)\left[W_1+(h+\theta)W_2\right]y(t+\theta)\mathrm{d}\theta,$$

is

$$\begin{aligned}\frac{\mathrm{d}R_4(t)}{\mathrm{d}t} &= y^T(t)\left[W_1+hW_2\right]y(t)-y^T(t-h)W_1y(t-h) \\ &\quad -\int_{-h}^{0} y^T(t+\theta)W_2 y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

Now, repeating the arguments applied in the proof of Theorem 2.4 we arrive at the desired expression for the derivative of $v(y_t)$. □

Let

$$\nu = \max_{\theta\in[0,h]} \|U(\theta)\|, \quad a = \|A_1\|.$$

Then the following estimates hold:

$$J_1(t) = 2y^T(t)\Delta_0^T U(0)y(t) \le 2\nu\rho_0 \|y(t)\|^2,$$
$$J_2(t) = 2y^T(t-h)\Delta_1^T U(0)y(t) \le \nu\rho_1 \left[\|y(t)\|^2 + \|y(t-h)\|^2\right],$$
$$J_3(t) = 2y^T(t)\Delta_0^T \int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta$$
$$\le h\nu\rho_0 a \|y(t)\|^2 + \nu\rho_0 a \int_{-h}^{0} \|y(t+\theta)\|^2 \mathrm{d}\theta,$$
$$J_4(t) = 2y^T(t-h)\Delta_1^T \int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta$$
$$\le h\nu\rho_1 a \|y(t-h)\|^2 + \nu\rho_1 a \int_{-h}^{0} \|y(t+\theta)\|^2 \mathrm{d}\theta.$$

From the preceding inequalities we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) \le -w(y_t) + \nu\left[2\rho_0 + h\rho_0 a + \rho_1\right]\|y(t)\|^2 + \nu\rho_1\left[1+ha\right]\|y(t-h)\|^2$$
$$+\nu\left[\rho_0+\rho_1\right]a\int_{-h}^{0}\|y(t+\theta)\|^2\mathrm{d}\theta,$$

and we arrive at the following statement.

Theorem 2.17. *Let system (2.1) be exponentially stable. Given positive-definite matrices W_0, W_1, W_2, system (2.42) remains exponentially stable for all Δ_0 and Δ_1 satisfying (2.43) if the following inequalities hold:*

1. $\lambda_{\min}(W_0) \ge \nu\left[2\rho_0 + h\rho_0 a + \rho_1\right]$,
2. $\lambda_{\min}(W_1) \ge \nu\rho_1\left[1+ha\right]$,
3. $\lambda_{\min}(W_2) \ge \nu\left[\rho_0+\rho_1\right]a$.

Remark 2.6. Theorem 2.17 remains true if we assume that the uncertain matrices Δ_0 and Δ_1 are continuous functions of t and x_t.

2.13 Notes and References

The first work dedicated to the construction of quadratic Lyapunov functionals with a given time derivative was that by Repin [63]. In this seminal contribution a quadratic functional of a general form was suggested. The time derivative of the functional was computed, and then, equating the derivative to the prescribed one, a system of equations for the matrices that define the functional was derived. The system includes a linear matrix partial differential equation, ordinary matrix differential equations, and algebraic relations between the matrices. Under some simplifying assumptions the system was reduced to a system of two matrix differential equations similar to that given by (2.21). Many essential elements needed for the computation of Lyapunov matrices, some in explicit form and some in implicit form, can be found there. Without a doubt this three-page contribution has had a profound impact on research in the area.

In the paper by Datko, [7], the main object was a presentation of an infinite-dimensional version of the Lyapunov–Krasovskii approach to the stability analysis of linear time-delay systems. In particular, the paper provides an interpretation from the operator point of view of the results given in [63].

The paper by Castelan and Infante [4] is dedicated to the following initial value problem:

$$\frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau} = AQ(\tau) + BQ^T(h-\tau), \quad Q\left(\frac{h}{2}\right) = K, \tag{2.44}$$

where K is a given $n \times n$ matrix. It is worth mentioning that the dynamic property in [63] was written in this form. In that paper, it was shown that for any given K the initial value problem admits a unique solution. An exhaustive analysis of the solution space of Eq. (2.44) is presented in the paper as well. The reader may find there interesting observations on the spectrum of an auxiliary system, similar to that presented in Sect. 2.12.3.

More interesting for us is the second paper by the same authors, [28]. There, for the first time, the three basic properties of Lyapunov matrices are explicitly indicated. Once again, following the tradition established in [63], the dynamic property was written in the form of Eq. (2.44). The symmetry property did not receive its due attention but was simply mentioned as a property of a matrix $\widetilde{Q}(\tau)$ similar to that of the improper integral (2.11). The algebraic property was introduced as a bridge connecting the matrices $Q(\tau)$ and $\widetilde{Q}(\tau)$. What is very important for us is that the paper discusses functionals similar to that of the complete type. For these functionals quadratic lower and upper bounds of the form given in Lemmas 2.12 and 2.13 were provided. The main goal of the paper was to demonstrate that functionals may be effectively applied to the computation of upper exponential estimates of the solutions of time-delay systems. Unfortunately, at one of the intermediate steps, namely, in the computation of an upper bound for a functional, the desired exponential estimate was explicitly used. The paper also contains a remark that for the case of exponentially stable systems functionals of the form (2.13) do not admit quadratic lower bounds.

The next serious breakthrough in this direction was made in the paper by Huang [26], where the existence of lower bounds for functionals of the form (2.13) is studied. The paper demonstrates, for the case of exponentially stable systems, that functionals admit local cubic lower bounds of the form

$$\alpha \|\varphi(0)\|^3 \leq v_0(\varphi), \quad \varphi \in C([-h,0],R^n), \text{ and } \|\varphi\|_h \leq H,$$

where α and H are two positive constants. It is important to note here that this result, as well as others in the present contribution, have been proven for a very general class of linear time-delay systems. This was probably the first paper to state explicitly that Lyapunov matrices are completely defined by three basic properties: dynamic, symmetry, and algebraic properties. The most important result presented in that paper is the existence theorem. The theorem states that if a time-delay system satisfies the Lyapunov condition, then for any symmetric matrix W there exists a corresponding Lyapunov matrix. Additionally, an explicit frequency domain expression for the Lyapunov matrix is given as well.

Theorem 2.8 was proven in [40]. The complete type functionals were introduced in [42], where some robustness bounds are derived as well. Complete type functionals are applied to the computation of exponential estimates of the solutions of system (2.1) in [38]. A brief account of the theory of Lyapunov matrices and functionals appears in [36].

A detailed account of the application of functionals of the form (2.13) to the computation of various quadratic performance indices can be found in an interesting book [55]; see also [8, 9, 14, 22, 27, 51, 52].

In the paper by Louisell [53] a relation between the spectrum of a time-delay system and that of an auxiliary delay-free system of matrix equations is established. That is, it is shown that any pure imaginary eigenvalue of a time-delay system is also an eigenvalue of an auxiliary system. The statement was obtained for the case of neutral type systems with one delay. In some sense the statement of Theorem 2.16 is a generalization of this important result.

The following open problem is one of the most important problems related to Lyapunov matrices and deserves to be mentioned here: Find the conditions of the exponential stability of system (2.1) expressed in the terms of a Lyapunov matrix $U(\tau)$, associated with a positive-definite matrix W. The first result in this direction was obtained by Mondie [56], where some necessary and sufficient conditions are derived for the case of scalar equations.

Chapter 3
Multiple Delay Case

In this chapter we address the case of retarded type linear time-delay systems with multiple delays. The fundamental matrix of such a system is defined. Then this matrix is used to derive an explicit expression for the solution of an initial value problem. Applying the scheme presented in the previous chapter, a general form of quadratic functionals with prescribed time derivatives along the solutions of the time-delay systems is obtained. It is shown that the functionals are defined by special matrix valued functions known as Lyapunov matrices for the system. A special system of matrix equations that defines Lyapunov matrices is derived. It is shown that the system admits a unique solution if and only if the spectrum of the time-delay system does not contain points arranged symmetrically with respect to the origin of the complex plane. This spectrum property is known as a Lyapunov condition. Two numerical schemes for the computation of Lyapunov matrices are presented. The first one is applicable to the case where time delays are multiple to a basic one. The other one makes it possible to compute approximate Lyapunov matrices in the case of general time delays. A measure that allows one to estimate the quality of the approximation is provided as well. Quadratic functionals of the complete type are defined, and several important applications of the functionals are presented in the final part of the chapter.

3.1 Preliminaries

We consider now a retarded type time-delay system of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{j=0}^{m} A_j x(t-h_j), \quad t \geq 0, \tag{3.1}$$

where A_j, $j = 0, 1, \ldots, m$, are given real $n \times n$ matrices and $0 = h_0 < h_1 < \cdots < h_m = h$ are time delays.

V.L. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*, Control Engineering, DOI 10.1007/978-0-8176-8367-2_3,

Definition 3.1 ([3]). The $n \times n$ matrix $K(t)$ is said to be the fundamental matrix for any system of the form (3.1) there is only one fundamental matrix

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = \sum_{j=0}^{m} K(t-h_j)A_j, \quad t \geq 0, \tag{3.2}$$

and $K(t) = 0_{n\times n}$ for $t < 0$, $K(0) = I$.

Theorem 3.1 ([3]). *Given an initial function $\varphi \in PC([-h,0],R^n)$, the following equality holds:*

$$x(t,\varphi) = K(t)\varphi(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t-\theta-h_j)A_j\varphi(\theta)\mathrm{d}\theta, \quad t \geq 0. \tag{3.3}$$

This equality is known as the Cauchy formula for system (3.1).

Proof. Assume that $t > 0$, and compute for $\xi \in (0,t)$ the partial derivative

$$\begin{aligned}\frac{\partial}{\partial \xi}\left[K(t-\xi)x(\xi,\varphi)\right] &= -\left[\sum_{j=0}^{m} K(t-\xi-h_j)A_j\right] x(\xi,\varphi) \\ &\quad + K(t-\xi)\left[\sum_{j=0}^{m} A_j x(\xi-h_j,\varphi)\right] \\ &= \sum_{j=1}^{m}\left[K(t-\xi)A_j x(\xi-h_j,\varphi) - K(t-\xi-h_j)A_j x(\xi,\varphi)\right].\end{aligned}$$

Integrating the preceding equality by ξ on the segment $[0,t]$, we obtain that the integral of the left-hand side is equal to

$$\int_0^t \left(\frac{\partial}{\partial \xi}\left[K(t-\xi)x(\xi,\varphi)\right]\right)\mathrm{d}\xi = x(t,\varphi) - K(t)\varphi(0).$$

Before computing the integral on the right-hand side we evaluate the integral

$$\begin{aligned}J &= \int_0^t \left[K(t-\xi)A_j x(\xi-h_j,\varphi) - K(t-\xi-h_j)A_j x(\xi,\varphi)\right]\mathrm{d}\xi \\ &= \int_{-h_j}^{t-h_j} K(t-\theta-h_j)A_j x(\theta,\varphi)\mathrm{d}\theta - \int_0^t K(t-\xi-h_j)A_j x(\xi,\varphi)\mathrm{d}\xi \\ &\quad + \int_{-h_j}^{0} K(t-\theta-h_j)A_j x(\theta,\varphi)\mathrm{d}\theta - \int_{t-h_j}^{t} K(t-\xi-h_j)A_j x(\xi,\varphi)\mathrm{d}\xi.\end{aligned}$$

Since $t-\xi-h_j<0$ for $\xi\in(t-h_j,t]$, matrix $K(t-\xi-h_j)=0_{n\times n}$, and the second integral on the right-hand side of the last equality disappears. We arrive at the equality

$$J=\int_{-h_j}^{0} K(t-\theta-h_j)A_jx(\theta,\varphi)\mathrm{d}\theta.$$

As a result, we conclude that

$$x(t,\varphi)-K(t)\varphi(0)=\sum_{j=1}^{m}\int_{-h_j}^{0} K(t-\theta-h_j)A_jx(\theta,\varphi)\mathrm{d}\theta.$$

The initial value condition $x(\theta,\varphi)=\varphi(\theta)$ for $\theta\in[-h,0]$ implies that the preceding equality coincides with (3.3). □

3.2 Quadratic Functionals

The statement of Theorem 2.3 remains valid for system (3.1), too. In this section we derive for system (3.1) functionals that satisfy the theorem conditions.

Like the case of single-delay systems we first define a quadratic functional $v_0(\varphi)$, $\varphi\in PC([-h,0],R^n)$, that satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t)=-x^T(t)Wx(t),\quad t\geq 0, \tag{3.4}$$

along the solutions of system (3.1). Here W is a given symmetric matrix.

If system (3.1) is exponentially stable, then the functional admits the integral representation

$$v_0(\varphi)=\int_0^{\infty} x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t,\quad \varphi\in PC([-h,0],R^n). \tag{3.5}$$

We replace $x(t,\varphi)$ under the integral sign on the right-hand side of (3.5) by Cauchy formula (3.3), and, after some straightforward manipulations, similar to that done in Sect. 2.5, we arrive at the following expression for the functional:

$$\begin{aligned}v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0)+\sum_{j=1}^{m}2\varphi^T(0)\int_{-h_j}^{0}U(-\theta-h_j)A_j\varphi(\theta)\mathrm{d}\theta\\ &\quad+\sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0}\varphi^T(\theta_1)A_k^T\left(\int_{-h_j}^{0}U(\theta_1+h_k-\theta_2-h_j)A_j\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1,\end{aligned} \tag{3.6}$$

where the matrix

$$U(\tau) = \int_0^\infty K^T(t) W K(t+\tau) \mathrm{d}t \tag{3.7}$$

is the Lyapunov matrix of system (3.1) associated with W.

Lemma 3.1. *Let system (3.1) be exponentially stable. The matrix $U(\tau)$ is continuous for $\tau \geq 0$.*

Proof. The matrix $K^T(t)WK(t+\tau)$ is continuous for $t \geq 0$ and $\tau \geq 0$. The exponential stability of system (3.1) implies that there exist $\gamma \geq 1$ and $\sigma > 0$ such that

$$\|K(t)\| \leq \gamma \mathrm{e}^{-\sigma t}, \quad t \geq 0.$$

Hence

$$\left\|K^T(t)WK(t+\tau)\right\| \leq \|K(t)\| \, \|W\| \, \|K(t+\tau)\| \leq \gamma^2 \|W\| \mathrm{e}^{-2\sigma t}.$$

Since the integral

$$\int_0^\infty \gamma^2 \|W\| \mathrm{e}^{-2\sigma t} \mathrm{d}t$$

converges, the improper integral on the right-hand side of (3.7) converges absolutely and uniformly with respect to τ, on the set $[0,\infty)$, and therefore the integral is a continuous function of the variable τ for $\tau \geq 0$. □

Lemma 3.2. *Let system (3.1) be exponentially stable. Lyapunov matrix (3.7) satisfies the following properties:*

1. Dynamic property:

$$\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = \sum_{j=0}^{m} U(\tau - h_j) A_j, \quad \tau \geq 0; \tag{3.8}$$

2. Symmetry property:

$$U(-\tau) = U^T(\tau); \tag{3.9}$$

3. Algebraic property:

$$\sum_{j=0}^{m} \left[U(-h_j) A_j + A_j^T U(h_j) \right] = -W. \tag{3.10}$$

Proof. *Dynamic property*: Let $\tau > 0$; then

$$\frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right] = \sum_{j=0}^{m} K^T(t)WK(t+\tau-h_j)A_j, \quad t \geq 0.$$

The exponential stability of system (3.1) implies that

$$\begin{aligned}\left\|\frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\right\| &\leq \sum_{j=0}^{m} \|K(t)\|\,\|W\|\,\left\|K(t+\tau-h_j)\right\|\,\left\|A_j\right\| \\ &\leq \sum_{j=0}^{m} \|W\|\,\left\|A_j\right\| \gamma^2 \mathrm{e}^{-\sigma(2t+\tau-h_j)} \\ &\leq \gamma^2 \mathrm{e}^{\sigma h}\|W\|\left(\sum_{j=0}^{m}\left\|A_j\right\|\right)\mathrm{e}^{-2\sigma t}, \quad t \geq 0.\end{aligned}$$

On the one hand, since the integral

$$\int_0^\infty \gamma^2 \mathrm{e}^{\sigma h}\|W\|\left(\sum_{j=0}^{m}\left\|A_j\right\|\right)\mathrm{e}^{-2\sigma t}\mathrm{d}t$$

converges, the integral

$$\int_0^\infty \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\mathrm{d}t$$

converges absolutely and uniformly with respect to τ on the set $[0,\infty)$, which implies the equality

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^\infty K^T(t)WK(t+\tau)\mathrm{d}t\right) \\ &= \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}, \quad \tau \geq 0.\end{aligned}$$

On the other hand, for $\tau > 0$

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right]\mathrm{d}t &= \int_0^\infty K^T(t)W\left[\sum_{j=0}^{m} K(t+\tau-h_j)A_j\right]\mathrm{d}t \\ &= \sum_{j=0}^{m} U(\tau-h_j)A_j,\end{aligned}$$

and we arrive at (3.8).

Symmetry property: The property follows directly from (3.7).

Algebraic property: To derive this property, we consider the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[K^T(t)WK(t)\right] = \left[\sum_{j=0}^{m} K(t-h_j)A_j\right]^T WK(t) + K^T(t)W\left[\sum_{j=0}^{m} K(t-h_j)A_j\right], \quad t \geq 0.$$

Now, integrating the preceding equality from 0 to ∞, we arrive at the equality

$$-W = \sum_{j=0}^{m}\left[A_j^T U^T(-h_j) + U(-h_j)A_j\right].$$

If we take into account (3.9), then it becomes evident that this equality coincides with (3.10). □

Lemma 3.3. *The first derivative of Lyapunov matrix (3.7) suffers discontinuity at* $\tau = 0$*, namely,*

$$U'(+0) - U'(-0) = -W.$$

Proof. The proof of the lemma is similar to that of Lemma 2.6. □

Theorem 3.2. *Let the matrix* $\widetilde{U}(\tau)$*,* $\tau \in [-h,h]$*, satisfy properties (3.8)–(3.10). If we define the functional* $\widetilde{v}_0(\varphi)$ *by formula (3.6), where the matrix* $U(\tau)$ *is replaced by the matrix* $\widetilde{U}(\tau)$*, then the functional is such that along the solutions of system (3.1) the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

Proof. Given a solution $x(t)$ of system (3.1),

$$\widetilde{v}_0(x_t) = \underbrace{x^T(t)\widetilde{U}(0)x(t)}_{R_0(t)} + \sum_{j=1}^{m}\underbrace{2x^T(t)\int_{-h_j}^{0}\widetilde{U}(-\theta-h_j)A_jx(t+\theta)\mathrm{d}\theta}_{R_j(t)}$$

$$+\sum_{k=1}^{m}\sum_{j=1}^{m}\underbrace{\int_{-h_k}^{0}x^T(t+\theta_1)A_k^T\left(\int_{-h_j}^{0}\widetilde{U}(\theta_1+h_k-\theta_2-h_j)A_jx(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1}_{R_{kj}(t)}.$$

Let us differentiate term by term the summands on the right-hand side of the preceding equality.

The time derivative of the first term is

$$\frac{\mathrm{d}R_0(t)}{\mathrm{d}t} = 2x^T(t)\widetilde{U}(0)\left[\sum_{j=0}^{m} A_j x(t-h_j)\right].$$

The time derivative of the term

$$R_j(t) = 2x^T(t)\int_{-h_j}^{0} \widetilde{U}(-\theta-h_j)A_j x(t+\theta)\mathrm{d}\theta = 2x^T(t)\int_{t-h_j}^{t} \widetilde{U}(-\xi+t-h_j)A_j x(\xi)\mathrm{d}\xi$$

is equal to

$$\begin{aligned}\frac{\mathrm{d}R_j(t)}{\mathrm{d}t} &= 2\left[\sum_{k=0}^{m} A_k x(t-h_k)\right]^T \int_{t-h_j}^{t} \widetilde{U}(-\xi+t-h_j)A_j x(\xi)\mathrm{d}\xi \\ &+ 2x^T(t)\widetilde{U}(-h_j)A_j x(t) - 2x^T(t)\widetilde{U}(0)A_j x(t-h_j) \\ &+ 2x^T(t)\int_{t-h_j}^{t}\left(\frac{\partial}{\partial t}\widetilde{U}(-\xi+t-h_j)\right)A_j x(\xi)\mathrm{d}\xi.\end{aligned}$$

The time derivative of the term

$$\begin{aligned}R_{kj}(t) &= \int_{-h_k}^{0} x^T(t+\theta_1)A_k^T\left(\int_{-h_j}^{0}\widetilde{U}(\theta_1+h_k-\theta_2-h_j)A_j x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\ &= \int_{t-h_k}^{t} x^T(\xi_1)A_k^T\left(\int_{t-h_j}^{t}\widetilde{U}(\xi_1+h_k-\xi_2-h_j)A_j x(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\end{aligned}$$

is written as

$$\begin{aligned}\frac{\mathrm{d}R_{kj}(t)}{\mathrm{d}t} &= x^T(t)A_k^T\int_{t-h_j}^{t}\widetilde{U}(t+h_k-\xi-h_j)A_j x(\xi)\mathrm{d}\xi \\ &\quad - x^T(t-h_k)A_k^T\int_{t-h_j}^{t}\widetilde{U}(t-\xi-h_j)A_j x(\xi)\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{t-h_k}^{t} x^T(\xi)A_k^T\widetilde{U}(\xi+h_k-t-h_j)\mathrm{d}\xi\right)A_jx(t)\\
&-\left(\int_{t-h_k}^{t} x^T(\xi)A_k^T\widetilde{U}(\xi+h_k-t)\mathrm{d}\xi\right)A_jx(t-h_j)\\
=\;& x^T(t)A_k^T\int_{t-h_j}^{t}\widetilde{U}(t+h_k-\xi-h_j)A_jx(\xi)\mathrm{d}\xi\\
&-x^T(t-h_k)A_k^T\int_{t-h_j}^{t}\widetilde{U}(t-\xi-h_j)A_jx(\xi)\mathrm{d}\xi\\
&+x^T(t)A_j^T\int_{t-h_k}^{t}\widetilde{U}(-\xi-h_k+t+h_j)A_kx(\xi)\mathrm{d}\xi\\
&-x^T(t-h_j)A_j^T\int_{t-h_k}^{t}\widetilde{U}(-\xi-h_k+t)A_kx(\xi)\mathrm{d}\xi.
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
\sum_{j=1}^{m}\frac{\mathrm{d}R_j(t)}{\mathrm{d}t} =\;& 2\left[\sum_{k=0}^{m}A_kx(t-h_k)\right]^T\sum_{j=1}^{m}\int_{t-h_j}^{t}\widetilde{U}(-\xi+t-h_j)A_jx(\xi)\mathrm{d}\xi\\
&+2x^T(t)\left[\sum_{j=1}^{m}\widetilde{U}(-h_j)A_j\right]x(t)-2x^T(t)\widetilde{U}(0)\left[\sum_{j=1}^{m}A_jx(t-h_j)\right]\\
&+2x^T(t)\sum_{j=1}^{m}\int_{t-h_j}^{t}\left(\frac{\partial}{\partial t}\widetilde{U}(-\xi+t-h_j)\right)A_jx(\xi)\mathrm{d}\xi
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{\mathrm{d}R_{kj}(t)}{\mathrm{d}t} =\;& 2x^T(t)\sum_{j=1}^{m}\int_{t-h_j}^{t}\left[\sum_{k=1}^{m}\widetilde{U}(-t-h_k+\xi+h_j)A_k\right]^T A_jx(\xi)\mathrm{d}\xi\\
&-2\left[\sum_{k=1}^{m}A_kx(t-h_k)\right]^T\sum_{j=1}^{m}\int_{t-h_j}^{t}\widetilde{U}(t-\xi-h_j)A_jx(\xi)\mathrm{d}\xi.
\end{aligned}
$$

Let us collect in the computed time derivatives all terms that have no integral factor. The sum of these terms is

$$\begin{aligned} S_1(t) &= 2x^T(t)\widetilde{U}(0)\left[\sum_{j=0}^{m} A_j x(t-h_j)\right] + 2x^T(t)\left[\sum_{j=1}^{m}\widetilde{U}(-h_j)A_j\right]x(t) \\ &\quad -2x^T(t)\widetilde{U}(0)\left[\sum_{j=1}^{m} A_j x(t-h_j)\right] \\ &= 2x^T(t)\left[\sum_{j=0}^{m}\widetilde{U}(-h_j)A_j\right]x(t) \\ &= x^T(t)\left[\left(\sum_{j=0}^{m}\widetilde{U}(-h_j)A_j\right) + \left(\sum_{j=0}^{m}\widetilde{U}(-h_j)A_j\right)^T\right]x(t). \end{aligned}$$

Since matrix $\widetilde{U}(\tau)$ satisfies the symmetry and algebraic properties, the sum

$$S_1(t) = -x^T(t)Wx(t).$$

Now we collect all terms that include an integral factor:

$$\begin{aligned} S_2(t) &= 2\left[\sum_{k=0}^{m} A_k x(t-h_k)\right]^T \sum_{j=1}^{m}\int_{t-h_j}^{t}\widetilde{U}(-\xi+t-h_j)A_j x(\xi)\mathrm{d}\xi \\ &\quad + 2x^T(t)\sum_{j=1}^{m}\int_{t-h_j}^{t}\frac{\partial\widetilde{U}(-\xi+t-h_j)}{\partial t}A_j x(\xi)\mathrm{d}\xi \\ &\quad + 2x^T(t)\sum_{j=1}^{m}\int_{t-h_j}^{t}\left[\sum_{k=1}^{m}\widetilde{U}(-t-h_k+\xi+h_j)A_k\right]^T A_j x(\xi)\mathrm{d}\xi \\ &\quad - 2\left[\sum_{k=1}^{m} A_k x(t-h_k)\right]^T \sum_{j=1}^{m}\int_{t-h_j}^{t}\widetilde{U}(t-\xi-h_j)A_j x(\xi)\mathrm{d}\xi \\ &= 2x^T(t)\sum_{j=1}^{m}\int_{t-h_j}^{t}\left[\sum_{k=0}^{m}\widetilde{U}(\tau-h_k)A_k - \frac{\mathrm{d}\widetilde{U}(\tau)}{\mathrm{d}\tau}\right]^T_{\tau=\xi-t+h_j} A_j x(\xi)\mathrm{d}\xi. \end{aligned}$$

In the integral

$$\int_{t-h_j}^{t}\left[\sum_{k=0}^{m}\widetilde{U}(\tau-h_k)A_k - \frac{\mathrm{d}\widetilde{U}(\tau)}{\mathrm{d}\tau}\right]^T_{\tau=\xi-t+h_j} A_j x(\xi)\mathrm{d}\xi,$$

the argument $\tau = \xi - t + h_j \in [0, h_j]$. The matrix $\widetilde{U}(\tau)$ satisfies the dynamic property (3.8), so we conclude that this integral is equal to zero. Therefore, $S_2(t) = 0_{n\times n}$, and we arrive at the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0. \qquad \square$$

Remark 3.1. In the proof of Theorem 3.2 we do not assume that system (3.1) is exponentially stable.

As in the case of single-delay systems, Theorem 3.2 lays the foundation for a new definition of the Lyapunov matrices.

Definition 3.2. Let the $n \times n$ matrix $U(\tau)$ satisfy Eq. (3.8). We say that it is a Lyapunov matrix *of system (3.1) associated with a symmetric matrix W* if it satisfies properties (3.9) and (3.10).

Because there are two definitions of a Lyapunov matrix, we must verify that they define the same matrix. During this verification we must keep in mind that (3.7) is valid for the exponentially stable systems only.

Theorem 3.3. *Let system (3.1) be exponentially stable; then matrix (3.7) is a unique solution of Eq. (3.8) that satisfies properties (3.9) and (3.10).*

Proof. Part 1: It was shown in Lemma 3.2 that matrix (3.7) satisfies Eq. (3.8) and properties (3.9) and (3.10). Thus, we need only prove the uniqueness of the solution. Assume that for a given W Eq. (3.8) admits two solutions, $U_1(\tau)$ and $U_2(\tau)$, that satisfy (3.9) and (3.10). Then we define two functionals $v_0^{(j)}(\varphi)$, $j = 1, 2$, of the form (3.6), the first one with the matrix $U_1(\tau)$, the second one with the matrix $U_2(\tau)$. By Theorem 3.2, we know that the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0^{(j)}(x_t) = -x^T(t)Wx(t), \quad j = 1, 2,$$

hold along the solutions of system (3.1). Hence, the difference $\Delta v(x_t) = v_0^{(2)}(x_t) - v_0^{(1)}(x_t)$ is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(x_t) = 0, \quad t \geq 0.$$

The preceding equality implies that for any initial function $\varphi \in PC([-h, 0], R^n)$ the following identity holds:

$$\Delta v(x_t(\varphi)) = \Delta v(\varphi), \quad t \geq 0.$$

Since system (3.1) is exponentially stable, $x_t(\varphi) \to 0_h$ as $t \to \infty$ and $\Delta v(x_t(\varphi)) \to 0$ as $t \to \infty$. This means that $\Delta v(\varphi) = 0$ for any initial function

$\varphi \in PC([-h,0],R^n)$ or, in explicit form,

$$0 = \varphi^T(0)\Delta U(0)\varphi(0) + \sum_{j=1}^{m} 2\varphi^T(0) \int_{-h_j}^{0} \Delta U(-\theta - h_j)A_j\varphi(\theta)\mathrm{d}\theta$$
$$+ \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \left(\int_{-h_j}^{0} \Delta U(\theta_1 + h_k - \theta_2 - h_j)A_j\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1. \tag{3.11}$$

Here the matrix $\Delta U(\tau) = U_2(\tau) - U_1(\tau)$ satisfies Eq. (3.8) and properties (3.9) and (3.10) with $W = 0_{n\times n}$.

Part 2: Now, given a vector $\gamma \in R^n$, consider the piecewise continuous initial function

$$\varphi^{(1)}(\theta) = \begin{cases} \gamma, & \theta = 0, \\ 0, & \theta \in [-h,0). \end{cases}$$

For this initial function equality (3.11) takes the form

$$0 = \Delta v(\varphi^{(1)}) = \gamma^T \Delta U(0)\gamma.$$

Since γ is an arbitrary vector and $\Delta U(0)$ is a symmetric matrix, the following equality holds:

$$\Delta U(0) = 0_{n\times n}. \tag{3.12}$$

Let us fix an index $i \in \{1,2\ldots,m\}$ and choose $\tau \in (h_{i-1},h_i]$and $\varepsilon > 0$ such that $-\tau+\varepsilon < -h_{i-1}$. Now, for given vectors $\gamma,\eta \in R^n$ we define the initial function

$$\varphi^{(2)}(\theta) = \begin{cases} \gamma, & \theta = 0, \\ \eta, & \theta \in [-\tau,-\tau+\varepsilon], \\ 0, & \text{for all other} \quad \theta \in [-h,0). \end{cases}$$

For this initial function, equality (3.11) is of the form

$$0 = \int_{-\tau}^{-\tau+\varepsilon} 2\gamma^T \left[\sum_{j=i}^{m} \Delta U(-h_j - \theta)A_j \right] \eta \mathrm{d}\theta$$
$$+ \sum_{k=i}^{m}\sum_{j=i}^{m} \int_{-\tau}^{-\tau+\varepsilon} \left(\int_{-\tau}^{-\tau+\varepsilon} \eta^T A_k^T \Delta U(\theta_1 + h_k - \theta_2 - h_j)A_j\eta \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1.$$

For sufficiently small $\varepsilon > 0$ the first integral on the right-hand side of the preceding equality is of the order ε, whereas the double integrals are of the order ε^2. This means that the preceding equality can be written as

$$0 = 2\varepsilon\gamma^T \left(\sum_{j=i}^{m} \Delta U(\tau - h_j)A_j \right) \eta + o(\varepsilon),$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to +0$. As γ and η are arbitrary vectors and ε can be made arbitrarily small, it follows from the preceding equality that

$$\sum_{j=i}^{m} \Delta U(\tau - h_j)A_j = 0_{n\times n}, \quad \text{for } \tau \in (h_{i-1}, h_i]. \tag{3.13}$$

Equality (3.13) holds for each $i = 1, 2, \ldots, m$. Since for $i = 1$ we have

$$\sum_{j=1}^{m} \Delta U(\tau - h_j)A_j = 0_{n\times n}, \quad \tau \in (0, h_1],$$

on the segment $(0, h_1]$ differential equation (3.8) for matrix $\Delta U(\tau)$ takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0, \quad \tau \in (0, h_1].$$

Now equality (3.12) implies that $\Delta U(\tau) = 0_{n\times n}$ for $\tau \in [0, h_1]$. On the interval $(h_1, h_2]$ Eq. (3.8) and equality (3.13) for $i = 2$ yield the delay equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0 + \Delta U(\tau - h_1)A_1, \quad \tau \in (h_1, h_2].$$

It has been shown that $\Delta U(\tau) = 0_{n\times n}$ on the interval $[0, h_1]$; therefore, $\Delta U(\tau) = 0_{n\times n}$ for $\tau \in (h_1, h_2]$ as well. Continuing this process we conclude that $\Delta U(\tau) = 0_{n\times n}$ for $\tau \in [0, h]$, i.e., $U_1(\tau) = U_2(\tau)$ for all $\tau \in [-h, h]$. And the unique solution of Eq. (3.8) that satisfies properties (3.9) and (3.10) is given by (3.7).

□

Theorem 3.4. *Given the symmetric matrices W_j, $j = 0, 1, \ldots, 2m$, let us define the functional*

$$w(\varphi) = \sum_{k=0}^{m} \varphi^T(-h_k)W_k\varphi(-h_k) + \sum_{j=1}^{m} \int_{-h_j}^{0} \varphi^T(\theta)W_{m+j}\varphi(\theta)\mathrm{d}\theta, \quad \varphi \in PC([-h, 0], R^n). \tag{3.14}$$

If there exists a Lyapunov matrix $U(\tau)$ associated with the matrix

$$W = W_0 + \sum_{j=1}^{m} (W_j + h_j W_{m+j}),$$

and $v_0(\varphi)$ is the functional (3.6) with this Lyapunov matrix, then the time derivative of the functional

$$v(\varphi) = v_0(\varphi) + \sum_{j=1}^{m} \int_{-h_j}^{0} \varphi^T(\theta)\left[W_j + (h_j+\theta)W_{m+j}\right]\varphi(\theta)\mathrm{d}\theta \tag{3.15}$$

along the solutions of system (3.1) is such that

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) = -w(x_t), \quad t \geq 0.$$

Proof. We know that

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0(x_t) = -x^T(t)Wx(t) = -x^T(t)\left[W_0 + \sum_{j=1}^{m}(W_j + h_j W_{m+j})\right]x(t).$$

A simple change of the integration variable provides the equality

$$\begin{aligned} Q_j(t) &= \int_{-h_j}^{0} x^T(t+\theta)\left[W_j + (h_j+\theta)W_{m+j}\right]x(t+\theta)\mathrm{d}\theta \\ &= \int_{t-h_j}^{t} x^T(\xi)\left[W_j + (h_j+\xi-t)W_{m+j}\right]x(\xi)\mathrm{d}\xi; \end{aligned}$$

therefore,

$$\begin{aligned} \sum_{j=1}^{m} \frac{\mathrm{d}}{\mathrm{d}t} Q_j(t) &= x^T(t)\left[\sum_{j=1}^{m}(W_j + h_j W_{m+j})\right]x(t) \\ &\quad - \sum_{j=1}^{m} x^T(t-h_j)W_j x(t-h_j) - \sum_{j=1}^{m} \int_{-h_j}^{0} x^T(t+\theta)W_{m+j}x(t+\theta)\mathrm{d}\theta. \end{aligned}$$

The theorem statement follows directly from the preceding expressions for the time derivatives. □

Definition 3.3. We say that functional (3.15) is of the complete type if the matrices W_j, $j = 0, 1, \ldots, 2m$, are positive definite.

Lemma 3.4. *Let system (3.1) be exponentially stable. Given the positive-definite matrices W_j, $j = 0, 1, \ldots, 2m$, there exists a positive constant α_1 such that the complete type functional (3.15) satisfies the inequality*

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h, 0], R^n).$$

Proof. To prove the inequality, we consider a modified functional of the form

$$\widetilde{v}(\varphi) = v(\varphi) - \alpha \|\varphi(0)\|^2 = v(\varphi) - \alpha \varphi^T(0)\varphi(0),$$

where α is a real parameter. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t), \quad t \geq 0.$$

Here

$$\begin{aligned}
\widetilde{w}(x_t) &= w(x_t) + 2\alpha x^T(t)\left[\sum_{j=0}^{m} A_j x(t-h_j)\right] \\
&\geq \left(x^T(t), x^T(t-h_1), \ldots, x^T(t-h_m)\right) L(\alpha) \begin{pmatrix} x(t) \\ x(t-h_1) \\ \vdots \\ x(t-h_m) \end{pmatrix}.
\end{aligned}$$

The matrix

$$L(\alpha) = \begin{pmatrix} W_0 & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & W_1 & \cdots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & W_m \end{pmatrix} + \alpha \begin{pmatrix} A_0 + A_0^T & A_1 & \cdots & A_m \\ A_1^T & 0_{n\times n} & \cdots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ A_m^T & 0_{n\times n} & \cdots & 0_{n\times n} \end{pmatrix}.$$

Because the matrices W_j, $j = 0, 1, \ldots, m$, are positive definite, there exists $\alpha_1 > 0$ such that the matrix $L(\alpha_1)$ is positive definite, and we conclude that

$$\widetilde{w}(x_t) \geq 0.$$

The exponential stability of system (3.1) makes it possible to present $\widetilde{v}(\varphi)$ in the form

$$\widetilde{v}(\varphi) = \int_0^{\infty} \widetilde{w}(x(t,\varphi))\mathrm{d}t \geq 0.$$

The preceding inequality demonstrates that $\alpha_1 \left\|\varphi(0)\right\|^2 \leq v(\varphi)$. □

Lemma 3.5. *Given the symmetric matrices $W_j,\ j = 0,1,\ldots,2m$, assume that system (3.1) admits a Lyapunov matrix associated with the matrix*

$$W = W_0 + \sum_{j=1}^{m} \left(W_j + h_j W_{m+j}\right).$$

Then there exists a positive constant α_2 such that the complete type functional (3.15) satisfies the inequalities

$$v(\varphi) \leq \alpha_2 \left\|\varphi\right\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

Proof. First we introduce the notations

$$\nu = \max_{\tau\in[0,h]} \left\|U(\tau)\right\|, \quad a_j = \left\|A_j\right\|, \quad j = 1,2,\ldots,m.$$

The following estimates hold for the terms of functional (3.15):

$$R_0 = \varphi^T(0)U(0)\varphi(0) \leq \nu \left\|\varphi\right\|_h^2;$$

now, for $j = 1,2,\ldots,m$,

$$R_j = 2\varphi^T(0) \int_{-h_j}^{0} U(-\theta - h_j)A_j\varphi(\theta)\mathrm{d}\theta \leq 2\nu a_j h_j \left\|\varphi\right\|_h^2$$

for $j,k \in \{1,2,\ldots,m\}$,

$$R_{kj} = \int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \left(\int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j)A_j\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$\leq \nu a_k a_j h_k h_j \left\|\varphi\right\|_h^2,$$

and, finally,

$$R = \int_{-h_j}^{0} \varphi^T(\theta)\left[W_j + (h_j+\theta)W_{m+j}\right]\varphi(\theta)\mathrm{d}\theta$$
$$\leq h_j\left(\left\|W_j\right\| + h_j\left\|W_{m+j}\right\|\right)\|\varphi\|_h^2.$$

As a result, we arrive at an upper estimation of the form

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2,$$

where

$$\alpha_2 = \nu\left(1+\sum_{j=1}^{m} a_j h_j\right)^2 + \sum_{j=1}^{m} h_j\left(\left\|W_j\right\| + h_j\left\|W_{m+j}\right\|\right). \qquad \square$$

Lemma 3.6. *Let system (3.1) be exponentially stable. Given the positive-definite matrices W_j, $j=0,1,\ldots,2m$, there exist positive constants β_j, $j=0,1,\ldots,m$, such that the complete type functional (3.15) admits a lower estimate of the form*

$$\beta_0\|\varphi(0)\|^2 + \sum_{j=1}^{m}\beta_j \int_{-h_j}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta \leq v(\varphi), \quad \varphi\in PC([-h,0],R^n). \tag{3.16}$$

Proof. To prove the inequality, we consider the functional

$$\widetilde{v}(\varphi) = v(\varphi) - \beta_0\|\varphi(0)\|^2 - \sum_{j=1}^{m}\beta_j\int_{-h_j}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.$$

Along the solutions of system (3.1) the functional is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t), \quad t\geq 0,$$

where

$$\widetilde{w}(x_t) = w(x_t) + 2\beta_0 x^T(t)\left[\sum_{k=0}^{m} A_k x(t-h_k)\right] + \sum_{j=1}^{m}\beta_j\left[\|x(t)\|^2 - \left\|x(t-h_j)\right\|^2\right]$$
$$\geq \left[x^T(t), x^T(t-h_1),\ldots,x^T(t-h_m)\right] Q(\beta_0,\beta_1,\ldots,\beta_m)\begin{bmatrix} x(t)\\ x(t-h_1)\\ \vdots \\ x(t-h_m)\end{bmatrix}.$$

Here the matrix

$$Q(\beta_0,\beta_1,\dots,\beta_m) = \begin{pmatrix} W_0 & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & W_1 & & 0_{n\times n} \\ \vdots & & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & W_m \end{pmatrix} + \beta_0 \begin{pmatrix} A_0 + A_0^T & A_1 & \cdots & A_m \\ A_1^T & 0_{n\times n} & & 0_{n\times n} \\ \vdots & & \ddots & \vdots \\ A_m^T & 0_{n\times n} & \cdots & 0_{n\times n} \end{pmatrix}$$

$$+ \begin{pmatrix} (\beta_1+\beta_2+\cdots+\beta_m)I & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & -\beta_1 I & & 0_{n\times n} \\ \vdots & & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & -\beta_m I \end{pmatrix}.$$

Since the matrices W_j, $j=0,1,\dots,m$, are positive definite, there exist positive constants β_j, $j=0,1,2,\dots,m$, for which the matrix $Q(\beta_0,\beta_1,\dots,\beta_m)$ is positive definite. For such a choice of the constants we have that

$$\widetilde{w}(x_t) \geq 0, \quad t \geq 0.$$

The preceding inequality implies that

$$\widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0, \quad \varphi \in PC([-h,0],R^n),$$

from where the desired inequality (3.16) follows immediately. □

Lemma 3.7. *Given the symmetric matrices W_j, $j=0,1,\dots,2m$, assume that system (3.1) admits a Lyapunov matrix associated with the matrix*

$$W = W_0 + \sum_{j=1}^{m} \left(W_j + h_j W_{m+j}\right).$$

Then for functional (3.15) there exist positive constants δ_j, $j=0,1,\dots,m$, such that the following inequality holds:

$$v(\varphi) \leq \delta_0 \|\varphi(0)\|^2 + \sum_{j=1}^{m} \delta_j \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n). \tag{3.17}$$

Proof. In the proof of the statement we make use of the notations introduced in the proof of Lemma 3.5.

It is evident that for the first term of functional (3.15) the following inequality holds:

$$R_0 = \varphi^T(0)U(0)\varphi(0) \leq \nu \|\varphi(0)\|^2.$$

The term

$$R_j = 2\varphi^T(0) \int_{-h_j}^{0} U(-h_j-\theta)A_j\varphi(\theta)\mathrm{d}\theta$$

admits the upper bound

$$R_j \leq \nu a_j h_j \|\varphi(0)\|^2 + \nu a_j \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

Now for the term

$$R_{kj} = \int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \left[\int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j)A_j\varphi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

we have

$$\begin{aligned}
R_{kj} &\leq \nu a_k a_j \left[\int_{-h_k}^{0} \|\varphi(\theta)\| \mathrm{d}\theta \right] \left[\int_{-h_j}^{0} \|\varphi(\theta)\| \mathrm{d}\theta \right] \\
&\leq \frac{1}{2}\nu a_k a_j \left[\int_{-h_k}^{0} \|\varphi(\theta)\| \mathrm{d}\theta \right]^2 + \frac{1}{2}\nu a_k a_j \left[\int_{-h_j}^{0} \|\varphi(\theta)\| \mathrm{d}\theta \right]^2 \\
&\leq \frac{1}{2}\nu a_k a_j h_k \int_{-h_k}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta + \frac{1}{2}\nu a_k a_j h_j \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.
\end{aligned}$$

Finally, we estimate the term

$$\begin{aligned}
R &= \int_{-h_j}^{0} \varphi^T(\theta) \left[W_j + (h+\theta)W_{m+j} \right] \varphi(\theta)\mathrm{d}\theta \\
&\leq \left(\|W_j\| + h_j \|W_{m+j}\| \right) \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.
\end{aligned}$$

Collecting the obtained estimations we conclude that

$$v(\varphi) \leq \delta_0 \|\varphi(0)\|^2 + \sum_{j=1}^{m} \delta_j \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where

$$\delta_0 = \nu \left(1 + \sum_{j=1}^{m} a_j h_j\right),$$

$$\delta_j = \nu a_j \left(1 + h_j \sum_{k=1}^{m} a_k\right) + \left(\|W_j\| + h_j \|W_{m+j}\|\right), \quad j = 1, 2, \ldots, m. \qquad \square$$

3.3 Lyapunov Matrices

Let us define the Laplace image of the fundamental matrix $K(t)$ of system (3.1),

$$H(s) = \int_0^\infty e^{-st} K(t) dt = \left(sI - \sum_{j=0}^{m} e^{-sh_j} A_j\right)^{-1}.$$

The function

$$f(s) = \det\left(sI - \sum_{j=0}^{m} e^{-sh_j} A_j\right) \tag{3.18}$$

is the characteristic function of the system. The zeros of the function form the spectrum

$$\Lambda = \{s \mid f(s) = 0\}$$

of the system. These zeros are poles of $H(s)$. If system (3.1) satisfies the Lyapunov condition, i.e., the set Λ does not contain a point s_0 such that $-s_0 \in \Lambda$, then the spectrum Λ can be split into two sets; the first one, $\Lambda^{(+)}$, includes the system eigenvalues with positive real part, whereas the second one, $\Lambda^{(-)}$, includes the system eigenvalues with negative real part.

3.3.1 Exponentially Stable Case

In the case where system (3.1) is exponentially stable, the set $\Lambda^{(+)}$ is empty. The Lyapunov matrix associated with a symmetric matrix W can be written as follows:

$$U(\tau) = \int_0^\infty K^T(t) W K(t+\tau) dt;$$

see (3.7). The fundamental matrix $K(t) = 0_{n\times n}$ for $t < 0$. Because the components of the matrix belong to the intersection of the spaces $L^1(-\infty,+\infty)$ and $L^2(-\infty,+\infty)$, the Fourier image [1] of the matrix is of the form

$$H(i\omega) = \int_{-\infty}^{\infty} K(t)e^{-i\omega t}dt = \int_{0}^{\infty} K(t)e^{-i\omega t}dt,$$

and the Fourier image of $K(t+\tau)$ is

$$\int_{-\infty}^{\infty} K(t+\tau)e^{-i\omega t}dt = e^{i\omega\tau}\int_{-\infty}^{\infty} K(t+\tau)e^{-i\omega(t+\tau)}dt = e^{i\omega\tau}H(i\omega).$$

Application of Plansherel's theorem [45] leads to the following expression for the Lyapunov matrix:

$$U(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)e^{-\tau\xi}d\xi,$$

where the notation V.P.(Valeur Principle) is defined as follows:

$$\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} F(\xi)d\xi = \lim_{r\to\infty}\frac{1}{2\pi i}\int_{-ir}^{ir} F(\xi)d\xi.$$

3.3.2 General Case

We present now an extension to the general case of the preceding expression for the Lyapunov matrices proven in [26]. But first we denote by

$$\text{Res}\{F(s),s_0\}$$

the residue of $F(s)$ at the point s_0.

Theorem 3.5. *Let system (3.1) satisfy the Lyapunov condition. Then for any symmetric matrix W, the matrix*

$$\begin{aligned}\widetilde{U}(\tau) = {} & \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)e^{-\tau\xi}d\xi \\ & + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{H^T(s)WH(-s)e^{-\tau s},s_0\right\} \\ & + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{H^T(-s)WH(s)e^{\tau s},s_0\right\} \end{aligned} \quad (3.19)$$

is a Lyapunov matrix of the system associated with W.

Proof. System (3.1) satisfies the Lyapunov condition, so the matrices $H(s)$ and $H(-s)$ have no poles on the imaginary axis of the complex plane. Let y be a real number; then for sufficiently large $|y|$ the matrix $H^T(iy)WH(-iy)\mathrm{e}^{-i\tau y}$ is of the order $|y|^{-2}$. This means that the improper integral on the right-hand side of (3.19) is well defined for all real τ.

Since for the case of exponentially stable systems the existence statement has already been proven (Theorem 3.3), we assume that system (3.1) is not exponentially stable. In this case $\Lambda^{(+)}$ is not empty and contains a finite set of complex numbers.

Part 1: Let us check first that matrix (3.19) satisfies symmetry property (3.9). By direct inspection of the matrices

$$
\begin{aligned}
\widetilde{U}(-\tau) &= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\mathrm{e}^{\tau\xi}\mathrm{d}\xi \\
&\quad + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(s)WH(-s)\mathrm{e}^{\tau s}, s_0\right\} \\
&\quad + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(-s)WH(s)\mathrm{e}^{-\tau s}, s_0\right\} \\
&\quad \langle\text{making a change in the integration variable } \lambda = -\xi\rangle \\
&= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(-\lambda)WH(\lambda)\mathrm{e}^{-\tau\lambda}\mathrm{d}\lambda \\
&\quad + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(s)WH(-s)\mathrm{e}^{\tau s}, s_0\right\} \\
&\quad + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(-s)WH(s)\mathrm{e}^{-\tau s}, s_0\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\widetilde{U}(\tau)\right]^T &= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(-\xi)WH(\xi)\mathrm{e}^{-\tau\xi}\mathrm{d}\xi \\
&\quad + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(-s)WH(s)\mathrm{e}^{-\tau s}, s_0\right\} \\
&\quad + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(s)WH(-s)\mathrm{e}^{\tau s}, s_0\right\},
\end{aligned}
$$

we conclude that they are equal.

Part 2: We address now algebraic property (3.10). To verify it, we compute the following matrix:

$$
\begin{aligned}
\mathcal{O} &= \sum_{j=0}^{m} \widetilde{U}(-h_j)A_j + \sum_{j=0}^{m} A_j^T \widetilde{U}(h_j) \\
&= \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} \left(H^T(\xi) W H(-\xi) \left[\sum_{j=0}^{m} \mathrm{e}^{h_j \xi} A_j \right] \right. \\
&\quad \left. + \left[\sum_{j=0}^{m} \mathrm{e}^{-h_j \xi} A_j \right]^T H^T(\xi) W H(-\xi) \right) \mathrm{d}\xi \\
&\quad + \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res} \left\{ H^T(s) W H(-s) \left[\sum_{j=0}^{m} \mathrm{e}^{h_j s} A_j \right], s_0 \right\} \\
&\quad + \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res} \left\{ H^T(-s) W H(s) \left[\sum_{j=0}^{m} \mathrm{e}^{-h_j s} A_j \right], s_0 \right\} \\
&\quad + \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res} \left\{ \left[\sum_{j=0}^{m} \mathrm{e}^{-h_j s} A_j \right]^T H^T(s) W H(-s), s_0 \right\} \\
&\quad + \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res} \left\{ \left[\sum_{j=0}^{m} \mathrm{e}^{h_j s} A_j \right]^T H^T(-s) W H(s), s_0 \right\}.
\end{aligned}
$$

It is a matter of simple calculation to check the identities

$$
H(s) \left[\sum_{j=0}^{m} \mathrm{e}^{-h_j s} A_j \right] = sH(s) - I
$$

and

$$
H(-s) \left[\sum_{j=0}^{m} \mathrm{e}^{h_j s} A_j \right] = -sH(-s) - I.
$$

Additionally,

$$\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} WH(-\xi)\mathrm{d}\xi = \langle \lambda = -\xi \rangle = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} WH(\lambda)\mathrm{d}\lambda.$$

Now, the matrix $\mathcal{O}$ has the form

$$\begin{aligned}\mathcal{O} = {} & \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[-H^T(\xi)W - WH(\xi)\right]\mathrm{d}\xi \\ & + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{-H^T(s)W, s_0\right\} + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{-H^T(s)W, s_0\right\} \\ & + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{-H^T(-s)W, s_0\right\} + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{-H^T(-s)W, s_0\right\}.\end{aligned}$$

The Lyapunov condition implies that no poles of the matrix $H(-s)$ lie in the set $\Lambda^{(+)}$, so the last two sums on the right-hand side of the preceding equality are zero matrices, and

$$\begin{aligned}\mathcal{O} = {} & -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W + WH(\xi)\right]\mathrm{d}\xi \\ & - \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{H^T(s)W + WH(s), s_0\right\}.\end{aligned} \tag{3.20}$$

By the residue theorem [1, 50],

$$\sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{H^T(s)W + WH(s), s_0\right\} = \lim_{r\to\infty}\frac{1}{2\pi i}\oint_{\Gamma(r)}\left[H^T(\xi)W + WH(\xi)\right]\mathrm{d}\xi,$$

where $\Gamma(r)$ is the Nyquist contour, consisting of the semicircle $C(r) = \left\{ re^{i\varphi} \,\middle|\, \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}$ and the segment $[ir, -ir]$ of the imaginary axis.
The contour integral

$$\begin{aligned}J(r) = {} & \frac{1}{2\pi i}\oint_{\Gamma(r)}\left[H^T(\xi)W + WH(\xi)\right]\mathrm{d}\xi \\ = {} & -\frac{1}{2\pi i}\int_{-ir}^{ir}\left[H^T(\xi)W + WH(\xi)\right]\mathrm{d}\xi \\ & + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[H^T(re^{i\varphi})W + WH(re^{i\varphi})\right] re^{i\varphi}\mathrm{d}\varphi;\end{aligned}$$

therefore

$$\lim_{r\to\infty} J(r) = -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W + WH(\xi)\right]\mathrm{d}\xi$$
$$+\lim_{r\to\infty}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[H^T(re^{i\varphi})W + WH(re^{i\varphi})\right]re^{i\varphi}\mathrm{d}\varphi.$$

Since $H(re^{i\varphi})re^{i\varphi}\to I$ as $r\to\infty$, uniformly by $\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$,

$$S = \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W + WH(s), s_0\right\}$$
$$= -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W + WH(\xi)\right]\mathrm{d}\xi + W.$$

Substituting the preceding equality into (3.20) we obtain that $\mathcal{O} = -W$. Thus matrix (3.19) satisfies (3.10).

Part 3: Let us address dynamic property (3.8). For a given $\tau > 0$ we compute the matrix

$$F(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{U}(\tau) - \sum_{j=0}^{m}\widetilde{U}(\tau - h_j)A_j$$
$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}H^T(\xi)WH(-\xi)\left[-\xi I - \sum_{j=0}^{m}\mathrm{e}^{h_j\xi}A_j\right]\mathrm{e}^{-\tau\xi}\mathrm{d}\xi$$
$$+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)WH(-s)\left[-sI - \sum_{j=0}^{m}\mathrm{e}^{h_js}A_j\right]\mathrm{e}^{-\tau s}, s_0\right\}$$
$$+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)WH(s)\left[sI - \sum_{j=0}^{m}\mathrm{e}^{-h_js}A_j\right]\mathrm{e}^{\tau s}, s_0\right\}$$
$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}H^T(\xi)W\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W\mathrm{e}^{-\tau s}, s_0\right\}$$
$$+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)W\mathrm{e}^{\tau s}, s_0\right\}.$$

Because the matrix $H(-s)$ has no poles in the set $\Lambda^{(+)}$, the sum

$$\sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(-s)We^{\tau s}, s_0\right\} = 0_{n\times n},$$

and we obtain

$$F(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)We^{-\tau\xi}d\xi + \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(s)We^{-\tau s}, s_0\right\}.$$

Once again, applying the residue theorem,

$$\begin{aligned} S_1 &= \sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{H^T(s)We^{-\tau s}, s_0\right\} = \lim_{r\to\infty}\frac{1}{2\pi i}\oint_{\Gamma(r)} H^T(\xi)We^{-\tau\xi}d\xi \\ &= -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)We^{-\tau\xi}d\xi + \lim_{r\to\infty}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H^T(re^{i\varphi})Wre^{i\varphi}e^{-\tau re^{i\varphi}}d\varphi. \end{aligned}$$

By Jordan's theorem [1, 50], the equality

$$\lim_{r\to\infty}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H^T(re^{i\varphi})Wre^{i\varphi}e^{-\tau re^{i\varphi}}d\varphi = 0_{n\times n}$$

holds for any $\tau > 0$, and we arrive at the conclusion that

$$F(\tau) = \frac{d}{d\tau}\widetilde{U}(\tau) - \sum_{j=0}^{m}\widetilde{U}(\tau-h_j)A_j = 0_{n\times n}, \quad \tau > 0.$$

The preceding equality remains valid when $\tau \to +0$, so matrix (3.19) satisfies property (3.8) for $\tau \geq 0$. This concludes the proof. □

We now show that the Lyapunov condition provides the uniqueness of the Lyapunov matrices. But first we recall the following statement; see [2].

Lemma 3.8. *Given a quasipolynomial*

$$f(t) = \sum_{j=1}^{m} e^{z_j t}p_j(t),$$

where z_j, $j=1,\dots,m$, are complex numbers such that $z_j \neq z_k$ for $j \neq k$ and $p_j(t)$, $j=1,\dots,m$, are polynomials, the identity $f(t)=0$, $t \geq 0$, implies $p_j(t) \equiv 0$, for $j=1,\dots,m$.

Theorem 3.6. *Let system (3.1) satisfy the Lyapunov condition. Then for any symmetric matrix W there exists a unique Lyapunov matrix of the system associated with W.*

Proof. Since for exponentially stable systems the statement was proven in Theorem 3.3, we restrict our attention here to the case where system (3.1) is not exponentially stable.

It was shown in Theorem 3.5 that for a given symmetric matrix W there exists a Lyapunov matrix $U(\tau)$ associated with W. Thus, in the remainder of the proof we concentrate on the uniqueness issue. Assume that there are two such matrices, $U_j(\tau)$, $j=1,2$. Then, by Theorem 3.2, the functionals $v_0^{(j)}(\varphi)$, $j=1,2$, computed by formula (3.6) with these matrices are such that the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0^{(j)}(x_t) = -w(x(t)), \quad t \geq 0; \quad j=1,2,$$

hold along the solutions of system (3.1). The difference $\Delta v(x_t) = v_0^{(2)}(x_t) - v_0^{(1)}(x_t)$ is such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta v(x_t) = 0, \quad t \geq 0.$$

We obtain that for any $\varphi \in PC([-h,0],R^n)$ the identity

$$\Delta v(x_t(\varphi)) = \Delta v(\varphi), \quad t \geq 0, \tag{3.21}$$

holds along the solution $x(t,\varphi)$ of system (3.1). This means that along any solution $x(t)$ of the system $\Delta v(x_t)$ maintains a constant value.

The rest of the proof is divided into two parts. In the first one we show that under the Lyapunov condition the functional $\Delta v(\varphi)$ is trivial, i.e., for any initial function $\varphi \in PC([-h,0],R^n)$ the following equality holds:

$$0 = \Delta v(\varphi). \tag{3.22}$$

In the other part we demonstrate that equality (3.22) implies that

$$\Delta U(\tau) = U_2(\tau) - U_1(\tau) = 0_{n \times n}, \quad \tau \in [0,h].$$

Part 1: Let $\chi > 0$ be an upper bound for the real parts of the system eigenvalues. Only a finite number of the system eigenvalues, $s_1, s_2, \dots, s_N$, lie in the vertical stripe

$$Z = \{ \ s \mid -\chi \leq \mathrm{Re(s)} \leq \chi \ \}$$

of the complex plane; see [3].

Every solution $x(t,\varphi)$ of the system can be presented as the sum

$$x(t,\varphi) = x^{(1)}(t) + x^{(2)}(t),$$

where $x^{(1)}(t)$ corresponds to the part of the system spectrum that lies in Z and $x^{(2)}(t)$ corresponds to the rest of the spectrum, which lies to the left of the vertical line $Re(s) = -\chi$.
The first term, $x^{(1)}(t)$, is a finite sum of the form

$$x^{(1)}(t) = \sum_{\ell=1}^{N} e^{s_\ell t} p^{(\ell)}(t), \tag{3.23}$$

where $p^{(\ell)}(t)$ is a polynomial with vector coefficients of degree less than the multiplicity of s_ℓ as a zero of the characteristic function (3.18), $\ell = 1,2,\dots,N$.
The second term, $x^{(2)}(t)$, admits an upper estimate of the form

$$\left\| x^{(2)}(t) \right\| \leq c e^{-(\chi+\varepsilon)t}, \quad t \geq 0. \tag{3.24}$$

Here c is a positive constant and ε is a small positive number.
The functional $\Delta v(x_t(\varphi))$ can be decomposed as follows:

$$\Delta v(x_t(\varphi)) = \Delta v\left(x_t^{(1)}\right) + 2\Delta z\left(x_t^{(1)}, x_t^{(2)}\right) + \Delta v\left(x_t^{(2)}\right),$$

where

$$\begin{aligned}
\Delta z\left(x_t^{(1)}, x_t^{(2)}\right) &= \left[x^{(1)}(t)\right]^T \Delta U(0) x^{(2)}(t) \\
&\quad + \left[x^{(1)}(t)\right]^T \sum_{j=1}^{m} \int_{-h_j}^{0} \Delta U(-h_j-\theta) A_j x^{(2)}(t+\theta) \mathrm{d}\theta \\
&\quad + \left[x^{(2)}(t)\right]^T \sum_{j=1}^{m} \int_{-h_j}^{0} \Delta U(-h_j-\theta) A_j x^{(1)}(t+\theta) \mathrm{d}\theta \\
&\quad + \sum_{k=1}^{m} \sum_{j=1}^{m} \int_{-h_k}^{0} \left[x^{(1)}(t+\theta_1)\right]^T \\
&\quad \times \left[\int_{-h_j}^{0} A_k^T \Delta U(\theta_1 + h_k - \theta_2 - h_j) A_j x^{(2)}(t+\theta_2) \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1.
\end{aligned}$$

On the one hand, since $x^{(1)}(t)$ and $x^{(2)}(t)$ satisfy system (3.1), $\Delta v(x_t^{(1)})$ and $\Delta v(x_t^{(2)})$ maintain constant values, and we obtain that $\Delta z(x_t^{(1)},x_t^{(2)})$ is also a constant. On the other hand, the choice of χ and inequality (3.24) guarantee that

$$\Delta v\left(x_t^{(2)}\right) \to 0, \text{ and } \Delta z\left(x_t^{(1)},x_t^{(2)}\right) \to 0, \quad \text{as } t \to \infty.$$

This means that

$$\Delta v\left(x_t^{(2)}\right) = 0, \text{ and } \Delta z\left(x_t^{(1)},x_t^{(2)}\right) = 0, \quad t \geq \infty,$$

and we arrive at the identity

$$\Delta v\left(x_t^{(1)}\right) = \Delta v(\varphi), \quad t \geq 0. \tag{3.25}$$

Here

$$\Delta v(x_t^{(1)}) = \underbrace{\left[x^{(1)}(t)\right]^T \Delta U(0) x^{(1)}(t)}_{I_1(t)} + \underbrace{2\left[x^{(1)}(t)\right]^T \sum_{j=1}^{m} \int_{-h_j}^{0} \Delta U(-h_j-\theta)A_j x^{(1)}(t+\theta)\mathrm{d}\theta}_{I_2(t)}$$

$$\underbrace{+\sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0}\left[x^{(1)}(t+\theta_1)\right]^T A_k^T \left[\int_{-h_j}^{0} \Delta U(\theta_1+h_k-\theta_2-h_j)A_j x^{(1)}(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1}_{I_3(t)}.$$

Expression (3.23) provides that

$$R_1(t) = \left[x^{(1)}(t)\right]^T \Delta U(0) x^{(1)}(t) = \sum_{\ell=1}^{N}\sum_{r=1}^{N} \mathrm{e}^{(s_\ell+s_r)t}\left[p^{(\ell)}(t)\right]^T \Delta U(0) p^{(r)}(t).$$

Here $\left[p^{(\ell)}(t)\right]^T \Delta U(0) p^{(r)}(t)$ is a polynomial in t of degree less than the sum of the multiplicities of s_ℓ and s_r.

The second term

$$R_2(t) = 2\sum_{j=1}^{m}\left[x^{(1)}(t)\right]^T \int_{-h_j}^{0} \Delta U(-h_j-\theta)A_j x^{(1)}(t+\theta)\mathrm{d}\theta$$

$$= 2\sum_{j=1}^{m}\left(\sum_{\ell=1}^{N}\sum_{r=1}^{N}\mathrm{e}^{(s_\ell+s_r)t}\left[p^{(\ell)}(t)\right]^T \int_{-h_j}^{0} \Delta U(-h_j-\theta)A_j \mathrm{e}^{s_r\theta} p^{(r)}(t+\theta)\mathrm{d}\theta\right).$$

It is easy to verify that

$$\left[p^{(\ell)}(t)\right]^T \int_{-h_j}^{0} \Delta U(-h_j-\theta)A_j \mathrm{e}^{s_r\theta} p^{(r)}(t+\theta)\mathrm{d}\theta$$

is also a polynomial in t of degree less than the sum of the multiplicities of s_ℓ and s_r.

The last term

$$\begin{aligned} R_3(t) &= \sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0} \left[x^{(1)}(t+\theta_1)\right]^T \\ &\quad \times \left[\int_{-h_j}^{0} A_k^T \Delta U(\theta_1+h_k-\theta_2-h_j)A_j x^{(1)}(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\ &= 2\sum_{k=1}^{m}\sum_{j=1}^{m}\left(\sum_{\ell=1}^{N}\sum_{r=1}^{N}\mathrm{e}^{(s_\ell+s_r)t}\int_{-h_k}^{0}\left[\mathrm{e}^{s_\ell\theta_1}p^{(\ell)}(t+\theta_1)\right]^T\right. \\ &\quad \left.\times\left(\int_{-h_j}^{0} A_k^T\Delta U(\theta_1+h_k-\theta_2-h_j)A_j\mathrm{e}^{s_r\theta_2}p^{(r)}(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1\right), \end{aligned}$$

and, once again, the function

$$\int_{-h_k}^{0}\left[\mathrm{e}^{s_\ell\theta_1}p^{(\ell)}(t+\theta_1)\right]^T\left(\int_{-h_j}^{0} A_k^T\Delta U(\theta_1+h_k-\theta_2-h_j)A_j\mathrm{e}^{s_r\theta_2}p^{(r)}(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$

is a polynomial in t of degree less than the sum of the multiplicities of s_ℓ and s_r. This analysis demonstrates that $\Delta v\left(x_t^{(1)}\right)$ is a function of the form

$$\Delta v\left(x_t^{(1)}\right) = \sum_{\ell=1}^{N}\sum_{r=1}^{N}\mathrm{e}^{(s_\ell+s_r)t}\alpha_{\ell r}(t),$$

where $\alpha_{\ell r}(t)$, $\ell, r = 1,2,\dots,N$, are polynomials. Now identity (3.25) takes the form

$$\sum_{\ell=1}^{N}\sum_{r=1}^{N}\mathrm{e}^{(s_\ell+s_r)t}\alpha_{\ell r}(t) = \mathrm{e}^{0t}\Delta v(\varphi), \quad t \geq 0.$$

Because no one of the sums $(s_\ell + s_r)$, $\ell, r \in \{1,2,\ldots,N\}$, is equal to zero, by Lemma 3.8, the preceding identity implies that $\Delta v(\varphi) = 0$.

We may summarize our analysis as follows. If system (3.1) satisfies the Lyapunov condition, then equality (3.22) holds for any initial function $\varphi \in PC([-h,0],R^n)$.

Part 2: In the proof of Theorem 3.3 it was demonstrated (see *Part 2* of the proof) how equality (3.22) implies the uniqueness of the Lyapunov matrix associated with W. This concludes the proof. □

We now consider the situation with Lyapunov matrices where the Lyapunov condition fails.

Theorem 3.7. *Suppose system (3.1) does not satisfy the Lyapunov condition. Then there exists a nontrivial Lyapunov matrix of the system associated with $W = 0_{n\times n}$.*

Proof. According to the theorem condition, there exists an eigenvalue s_0 of system (3.1) such that $-s_0$ is also an eigenvalue of the system. The characteristic matrix of system (3.1) is of the form

$$G(s) = sI - \sum_{k=0}^{m} \mathrm{e}^{-h_k s} A_k.$$

Since matrices $G(s_0)$ and $G(-s_0)$ are singular, there exist nonzero vectors γ and μ such that

$$\gamma^T G(s_0) = 0, \quad \mu^T G(-s_0) = 0.$$

Let us define the following matrix:

$$U_0(\tau) = \mu\gamma^T \mathrm{e}^{s_0\tau} + \gamma\mu^T \mathrm{e}^{-s_0\tau}.$$

It is evident that the matrix is nontrivial. We verify that $U_0(\tau)$ is a Lyapunov matrix associated with $W = 0_{n\times n}$. First we note that the matrix satisfies the symmetry property

$$U_0(-\tau) = \mu\gamma^T \mathrm{e}^{-s_0\tau} + \gamma\mu^T \mathrm{e}^{s_0\tau} = U_0^T(\tau).$$

Then we verify that the matrix satisfies the dynamic property

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} U_0(\tau) - \sum_{k=0}^{m} U_0(\tau - h_k) A_k &= \mu\gamma^T G(s_0) \mathrm{e}^{s_0\tau} + \gamma\mu^T G(-s_0) \mathrm{e}^{-s_0\tau} \\ &= 0_{n\times n}. \end{aligned}$$

According to Lemma 3.3, the algebraic property can be written as

$$U'(+0) - U'(-0) = -W.$$

Since the matrix $U_0(\tau)$ is differentiable at $\tau = 0$, we conclude that

$$U_0'(+0) - U_0'(-0) = 0_{n\times n}.$$

This concludes the proof. □

Corollary 3.1. *If $U_0(\tau)$ is a complex valued matrix, then matrices $X(\tau) = \mathrm{Re}\{U_0(\tau)\}$ and $Y(\tau) = \mathrm{Im}\{U_0(\tau)\}$ are real Lyapunov matrices of system (3.1) associated with $W = 0_{n\times n}$.*

Corollary 3.2. *Let system (3.1) admit an eigenvalue s_0 such that $-s_0$ is also an eigenvalue of the system. If there exists a Lyapunov matrix $U(\tau)$ associated with a symmetric matrix W, then for any constant α the matrix $U(\tau) + \alpha U_0(\tau)$ is also a Lyapunov matrix associated with W.*

Theorem 3.8. *Suppose system (3.1) does not satisfy the Lyapunov condition. Then there exists a symmetric matrix W such that there is no Lyapunov matrix associated with W.*

Proof. According to the theorem condition, there exists an eigenvalue s_0 of system (3.1) such that $-s_0$ is also an eigenvalue of the system. System (3.1) admits two solutions of the form

$$x^{(1)}(t) = \mathrm{e}^{s_0 t}\gamma, \qquad x^{(2)}(t) = \mathrm{e}^{-s_0 t}\mu,$$

where γ and μ are nontrivial vectors. Assume by contradiction that for any symmetric matrix W there is a Lyapunov matrix associated with W. According to Lemma 2.9, there exists a symmetric matrix W_0 such that $\gamma^T W_0 \mu \neq 0$. Let $U(\tau)$ be a Lyapunov matrix associated with W_0. Let us define the bilinear functional

$$\begin{aligned} z(\varphi,\psi) &= \varphi^T(0)U(0)\psi(0) + \sum_{j=1}^{m} \varphi^T(0) \int_{-h_j}^{0} U(-h_j-\theta)A_j\psi(\theta)\mathrm{d}\theta \\ &\quad + \sum_{j=1}^{m} \left(\int_{-h_j}^{0} \varphi^T(\theta)A_j^T U(h_j+\theta)\mathrm{d}\theta \right)\psi(0) \\ &\quad + \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \left[\int_{-h_j}^{0} U(\theta_1+h_k-\theta_2-h_j)A_j\psi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1. \end{aligned}$$

Given two solutions of system (3.1), $x(t,\varphi)$ and $x(t,\psi)$, one can verify by direct calculation that

$$\frac{\mathrm{d}}{\mathrm{d}t} z(x_t(\varphi),x_t(\psi)) = -x^T(t,\varphi)W_0 x(t,\psi).$$

In particular, for the solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}z(x_t^{(1)},x_t^{(2)}) = -\left[x^{(1)}(t)\right]^T W_0 x^{(2)}(t) = -\gamma^T W_0 \mu \neq 0. \tag{3.26}$$

On the other hand, direct substitution of these solutions into the bilinear functional yields

$$z(x_t^{(1)},x_t^{(2)}) = \gamma^T \left[U(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} \left(U(-h_j-\theta)A_j \mathrm{e}^{-s_0\theta} + A_j^T U(h_j+\theta)\mathrm{e}^{s_0\theta} \right) \mathrm{d}\theta \right.$$
$$\left. + \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} \mathrm{e}^{-s_0\theta_1} \left(\int_{-h_j}^{0} \mathrm{e}^{s_0\theta_2} A_k^T U(\theta_1+h_k-\theta_2-h_j)A_j \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \right] \mu.$$

Observe that the matrix in the square brackets does not depend on t, therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}z(x_t^{(1)},x_t^{(2)}) = 0.$$

But this contradicts (3.26). Thus, no Lyapunov matrix associated with the selected W_0 exists. □

3.4 Computational Schemes

3.4.1 Semianalytic Method

Here a system with one basic delay $\mathfrak{h} \geq 0$ is considered, i.e., we study the case where

$$h_k = k\mathfrak{h}, \quad k = 1,\ldots,m.$$

Delay matrix Eq. (3.8) is now of the form

$$\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = \sum_{k=0}^{m} U(\tau - k\mathfrak{h})A_k, \quad \tau \geq 0. \tag{3.27}$$

Let us define for $\xi \in [0,\mathfrak{h}]$ the auxiliary matrices

$$Y_j(\xi) = U(j\mathfrak{h}+\xi), \quad j = -m,-m+1,\ldots,0,\ldots,m-1. \tag{3.28}$$

Lemma 3.9. *Let $U(\tau)$ be a Lyapunov matrix associated with W. Then for $\xi \in [0,\mathfrak{h}]$ auxiliary matrices (3.28) satisfy the system of linear delay-free matrix differential equations*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}Y_j(\xi) = \sum\limits_{k=0}^{m} Y_{j-k}(\xi)A_k, \quad j=0,1,\dots,m-1, \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi}Y_{-j}(\xi) = -\sum\limits_{k=0}^{m} A_k^T Y_{-j+k}(\xi), \quad j=1,2,\dots,m \end{cases} \tag{3.29}$$

and the boundary value conditions

$$Y_j(0) = Y_{j-1}(\mathfrak{h}), \quad j=-m+1,-m+2,\dots,0,\dots,m-1, \tag{3.30}$$

$$-W = \sum_{k=0}^{m-1}\left[Y_{-k}(0)A_k + A_k^T Y_k(0)\right] + Y_{m-1}(\mathfrak{h})A_m + A_m^T Y_{-m}(0). \tag{3.31}$$

Proof. First, we observe that for $j \in \{0,1,\dots,m-1\}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}Y_j(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi}U(j\mathfrak{h}+\xi) = \sum_{k=0}^{m} U(j\mathfrak{h}+\xi-k\mathfrak{h})A_k \\ &= \sum_{k=0}^{m} Y_{j-k}(\xi)A_k. \end{aligned}$$

Now, assume that $j \in \{1,2,\dots,m\}$; then

$$Y_{-j}(\xi) = U(-j\mathfrak{h}+\xi) = U^T(j\mathfrak{h}-\xi),$$

and for $\xi \in [0,\mathfrak{h}]$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}Y_{-j}(\xi) &= \left[\frac{\mathrm{d}}{\mathrm{d}\xi}U(j\mathfrak{h}-\xi)\right]^T = -\left[\sum_{k=0}^{m} U(j\mathfrak{h}-\xi-k\mathfrak{h})A_k\right]^T \\ &= -\sum_{k=0}^{m} A_k^T U(-j\mathfrak{h}+\xi+k\mathfrak{h}) \\ &= -\sum_{k=0}^{m} A_k^T Y_{-j+k}(\xi). \end{aligned}$$

Conditions (3.30) follow directly from the definition of matrices (3.28). The last boundary value condition is algebraic property (3.10) written in the terms of the auxiliary matrices. □

Theorem 3.9. *Assume that the boundary value problem (3.29)–(3.31) admits a solution*

$$\{Y_{m-1}(\xi), Y_{m-2}(\xi), \dots, Y_0(\xi), \dots, Y_{-m}(\xi)\},\ \xi \in [0,\mathfrak{h}].$$

Then the matrix $U(\tau)$, defined for $\tau \in [0, m\mathfrak{h}]$ by the expressions

$$U(j\mathfrak{h}+\xi) = \frac{1}{2}\left[Y_j(\xi) + Y_{-j-1}^T(\mathfrak{h}-\xi)\right], \; \xi \in [0,\mathfrak{h}], \quad j = 0,1,\ldots,m-1, \quad (3.32)$$

is a Lyapunov matrix associated with W if we extend it to $[-m\mathfrak{h}, 0)$ by setting $U(-\tau) = U^T(\tau)$ for $\tau \in (0, m\mathfrak{h}]$.

Proof. We prove that the matrices

$$\widetilde{Y}_j(\xi) = Y_{-j-1}^T(\mathfrak{h}-\xi), \quad j = -m, -m+1, \ldots, 0, \ldots, m-1,$$

satisfy boundary value problem (3.29)–(3.31) as well. To this end, we verify first that the matrices satisfy system (3.29). For $j \in \{0, 1, \ldots, m-1\}$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{Y}_j(\xi) &= \left[\frac{\mathrm{d}}{\mathrm{d}\xi}Y_{-j-1}(\mathfrak{h}-\xi)\right]^T = -\left[-\sum_{k=0}^{m} A_k^T Y_{-j-1+k}(\mathfrak{h}-\xi)\right]^T \\ &= \sum_{k=0}^{m} \widetilde{Y}_{j-k}(\xi) A_k,\end{aligned}$$

and for $j \in \{1, 2, \ldots, m\}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{Y}_{-j}(\xi) &= \left[\frac{\mathrm{d}}{\mathrm{d}\xi}Y_{j-1}(\mathfrak{h}-\xi)\right]^T = -\left[\sum_{k=0}^{m} Y_{j-1-k}(\mathfrak{h}-\xi) A_k\right]^T \\ &= -\sum_{k=0}^{m} A_k^T \widetilde{Y}_{-j+k}(\xi).\end{aligned}$$

Then we verify conditions (3.30)

$$\begin{aligned}\widetilde{Y}_j(0) - \widetilde{Y}_{j-1}(\mathfrak{h}) &= \left[Y_{-j-1}(\mathfrak{h}) - Y_{-j}(0)\right]^T = 0_{n\times n}, \\ j &= -m+1, -m+2, \ldots, 0, \ldots, m-1.\end{aligned}$$

And, finally, we address condition (3.31). If we take into account the equalities

$$\widetilde{Y}_j(0) = Y_{-j-1}^T(\mathfrak{h}) = Y_{-j}^T(0), \quad j = -m+1, -m+2, \ldots, m-1,$$

and

$$\widetilde{Y}_{-m}(0) = Y_{m-1}^T(\mathfrak{h}), \quad \widetilde{Y}_{m-1}(\mathfrak{h}) = Y_{-m}^T(0),$$

then

$$\begin{aligned}
\widetilde{R} &= \sum_{k=0}^{m-1} \left[\widetilde{Y}_{-k}(0)A_k + A_k^T \widetilde{Y}_k(0) \right] + \widetilde{Y}_{m-1}(\mathfrak{h})A_m + A_m^T \widetilde{Y}_{-m}(0) \\
&= \sum_{k=0}^{m-1} \left[Y_k^T(0)A_k + A_k^T Y_{-k}^T(0) \right] + Y_{-m}^T(0)A_m + A_m^T Y_{m-1}^T(\mathfrak{h}) \\
&= \left(\sum_{k=0}^{m-1} \left[A_k^T Y_k(0) + Y_{-k}(0)A_k \right] + A_m^T Y_{-m}(0) + Y_{m-1}(\mathfrak{h})A_m \right)^T \\
&= -W^T = -W.
\end{aligned}$$

Using the matrices $\widetilde{Y}_j(\xi)$, $j = -m, -m+1, \ldots, 0, \ldots, m-1$ we present the matrix $U(\tau)$, $\tau \in [0, m\mathfrak{h}]$, in the form

$$U(j\mathfrak{h} + \xi) = \frac{1}{2} \left[Y_j(\xi) + \widetilde{Y}_j(\xi) \right], \quad \xi \in [0, \mathfrak{h}].$$

Now we check that the matrix $U(\tau)$ defined by (3.32) satisfies Definition 3.2. We start with symmetry property (3.9). Since we define this matrix on $[-m\mathfrak{h}, 0)$ by setting $U(-\tau) = U^T(\tau)$, we only have to show that the matrix $U(0)$ is symmetric. It follows from (3.32) and (3.30) that

$$U(0) = \frac{1}{2} \left[Y_0(0) + Y_{-1}^T(\mathfrak{h}) \right] = \frac{1}{2} \left[Y_0(0) + Y_0^T(0) \right] = U^T(0).$$

Then we verify algebraic property (3.10). Here we observe that

$$U(j\mathfrak{h}) = \frac{1}{2} \left[Y_j(0) + \widetilde{Y}_j(0) \right], \quad j = -m, -m+1, \ldots, m-1,$$

and

$$U(m\mathfrak{h}) = \frac{1}{2} \left[Y_{m-1}(\mathfrak{h}) + \widetilde{Y}_{m-1}(\mathfrak{h}) \right].$$

Substituting the preceding expressions for the matrices $U(j\mathfrak{h})$ into (3.10) we obtain that

$$\begin{aligned}
J &= \sum_{j=0}^{m} \left[A_j^T U(j\mathfrak{h}) + U(-j\mathfrak{h})A_j \right] \\
&= \frac{1}{2} \left(\sum_{j=0}^{m-1} \left[A_j^T Y_j(0) + Y_{-j}(0)A_j \right] + A_m^T Y_{m-1}(\mathfrak{h}) + Y_{-m}(0)A_m \right)
\end{aligned}$$

$$+\frac{1}{2}\left(\sum_{j=0}^{m-1}\left[A_j^T\widetilde{Y}_j(0)+\widetilde{Y}_{-j}(0)A_j\right]+A_m^T\widetilde{Y}_{m-1}(\mathfrak{h})+\widetilde{Y}_{-m}(0)A_m\right)$$
$$=-\frac{1}{2}W-\frac{1}{2}W=-W.$$

Finally, we show that matrix (3.32) satisfies for $\tau\in[0,m\mathfrak{h}]$ dynamic property (3.27). We have already seen that for $j\in\{0,1,\dots,m-1\}$ the following expression holds:

$$U(j\mathfrak{h}+\xi)=\frac{1}{2}\left[Y_j(\xi)+\widetilde{Y}_j(\xi)\right],\quad \xi\in[0,\mathfrak{h}].$$

Now for $j\in\{1,2,\dots,m\}$ we have

$$\begin{aligned}U(-j\mathfrak{h}+\xi)&=U^T(j\mathfrak{h}-\xi)=U^T((j-1)\mathfrak{h}+\mathfrak{h}-\xi)\\&=\frac{1}{2}\left[Y_{j-1}(\mathfrak{h}-\xi)+\widetilde{Y}_{j-1}(\mathfrak{h}-\xi)\right]^T\\&=\frac{1}{2}\left[\widetilde{Y}_{-j}(\xi)+Y_{-j}(\xi)\right],\quad \xi\in[0,\mathfrak{h}].\end{aligned}$$

Keeping in mind these expressions we compute the first derivative of the matrix $U(\tau)$. Let $\tau=j\mathfrak{h}+\xi$, where $j\in\{0,1,\dots,m-1\}$ and $\xi\in(0,\mathfrak{h})$; then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)&=\frac{\mathrm{d}}{\mathrm{d}\xi}U(j\mathfrak{h}+\xi)=\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2}\left[Y_j(\xi)+\widetilde{Y}_j(\xi)\right]\right)\\&=\sum_{k=0}^{m}\frac{1}{2}\left[Y_{j-k}(\xi)+\widetilde{Y}_{j-k}(\xi)\right]A_k=\sum_{k=0}^{m}U(j\mathfrak{h}+\xi-k\mathfrak{h})A_k\\&=\sum_{k=0}^{m}U(\tau-k\mathfrak{h})A_k.\end{aligned}$$

It is left to check that for $j\in\{1,2,\dots,m\}$ the right-hand-side limit of the derivative of $U(\tau)$ at $\tau=j\mathfrak{h}$ coincides with the left-hand-side limit of the derivative at the point. The right-hand-side limit is equal to

$$\sum_{k=0}^{m}\frac{1}{2}\left[Y_{j-k}(0)+\widetilde{Y}_{j-k}(0)\right]A_k,$$

whereas the left-hand-side limit is

$$\sum_{k=0}^{m}\frac{1}{2}\left[Y_{j-1-k}(\mathfrak{h})+\widetilde{Y}_{j-1-k}(\mathfrak{h})\right]A_k.$$

The equality of these two limits is a direct consequence of the fact that the matrices $Y_j(\xi)$ and $\widetilde{Y}_j(\xi)$ satisfy boundary value conditions (3.30). □

Corollary 3.3. *If boundary value problem (3.29)–(3.31) admits a unique solution*

$$\{Y_{m-1}(\xi), Y_{m-2}(\xi), \dots, Y_0(\xi), \dots, Y_{-m}(\xi)\}, \quad \xi \in [0, \mathfrak{h}],$$

then there exists a unique Lyapunov matrix $U(\tau)$ associated with the matrix W, and the matrix is defined on $[0, m\mathfrak{h}]$ by the equalities

$$U(j\mathfrak{h} + \xi) = Y_j(\xi), \quad \xi \in [0, \mathfrak{h}], \quad j = 0, 1, \dots, m-1.$$

Remark 3.2. The size of delay-free system (3.29) may become very large even for a reasonably simple time-delay system. To illustrate this, let us consider a system with two delays, $h_1 = 0.571$ and $h_2 = 1$. The basic delay here is equal to $\mathfrak{h} = 0.001$. Therefore, in this case the number of auxiliary matrices in system (3.29) is equal to 2,000.

3.4.2 Scalar Equation

We consider a scalar time-delay equation of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{j=0}^{m} a_j x(t - j\mathfrak{h}), \quad t \geq 0, \tag{3.33}$$

where a_j, $j = 0, 1, \dots, m$, are real coefficients and $\mathfrak{h} > 0$ is a basic time delay.

For Eq. (3.33) Lyapunov matrices are scalar functions, and we call them scalar Lyapunov functions. Here W is a 1×1 matrix, and in this section we will use the lowercase letter μ instead of W. According to Definition 3.2, a scalar Lyapunov function associated with a given scalar value μ is a function that satisfies the following three properties:

1. Dynamic property:

$$\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} = \sum_{j=0}^{m} a_j u(\tau - j\mathfrak{h}), \quad \tau \geq 0;$$

2. Symmetry property:

$$u(-\tau) = u(\tau), \quad \tau \geq 0;$$

3. Algebraic property:

$$2\sum_{j=0}^{m} a_j u(j\mathfrak{h}) = -\mu.$$

Remark 3.3. If $u(\tau)$ is a scalar Lyapunov function associated with $\mu = 1$, then $\alpha u(\tau)$ is a scalar Lyapunov function associated with $\mu = \alpha$. Therefore, in the scalar case it is sufficient to compute the scalar Lyapunov function $u(\tau)$ associated with $\mu = 1$. The algebraic property for this function is of the form

$$\sum_{j=0}^{m} a_j u(j\mathfrak{h}) = -\frac{1}{2}. \tag{3.34}$$

Let us introduce for $\xi \in [0, \mathfrak{h}]$ the auxiliary functions

$$y_j(\xi) = u(j\mathfrak{h} + \xi), \quad j = -m, -m+1, \dots, 0, 1, \dots, m-1.$$

Delay-free system (3.29) is now of the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix} y_{m-1} \\ \vdots \\ y_0 \\ y_{-1} \\ \vdots \\ y_{-m} \end{pmatrix} = L \begin{pmatrix} y_{m-1} \\ \vdots \\ y_0 \\ y_{-1} \\ \vdots \\ y_{-m} \end{pmatrix}, \tag{3.35}$$

where

$$L = \begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_m & & & \\ 0 & a_0 & a_1 & \cdots & \cdots & a_m & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & a_m \\ -a_m & -a_{m-1} & \cdots & \cdots & -a_0 & & & \\ & -a_m & -a_{m-1} & \cdots & \cdots & -a_0 & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & -a_m & -a_{m-1} & \cdots & \cdots & -a_0 \end{pmatrix},$$

and boundary value conditions (3.30) and (3.31) are written as

$$y_j(0) = y_{j-1}(\mathfrak{h}), \quad j = -m+1, -m+2, \dots, 0, \dots, m-1,$$
$$\sum_{k=0}^{m-1} a_k y_k(0) + a_m y_{m-1}(\mathfrak{h}) = -\frac{1}{2}.$$

Remark 3.4. The matrix L in system (3.35) is the resultant matrix of the polynomials

$$p_1(s) = a_0 s^m + a_1 s^{m-1} + \dots + a_m,$$
$$p_2(s) = -a_m s^m - a_{m-1} s^{m-1} - \dots - a_0 = -s^m p_1\left(s^{-1}\right).$$

3.4.3 Numerical Scheme

Let us return to the case of general delays. The dynamic equation for the Lyapunov matrices is now of the form (3.8). We propose a computational scheme that consists of two stages. At the first one a piecewise linear approximation of the initial condition for the Lyapunov matrix is computed. Then, at the second stage, this initial condition is used for the computation of the approximate Lyapunov matrix as a solution of Eq. (3.8).

Initial Conditions

In this subsection we present an algorithm for the computation of a continuous piecewise linear approximation of the initial condition for a Lyapunov matrix $U(\tau)$.

First, we divide the interval $[-h,0]$ into N equal segments $[-(k+1)r,-kr]$, $k=0,1,\dots,N-1$, where $r=\frac{h}{N}$. Then we introduce $N+1$ auxiliary matrices Φ_j, $j=0,1,\dots,N$, and define the continuous piecewise linear matrix valued function

$$\Phi(\theta)=\begin{cases}\left(1+\frac{\theta+jr}{r}\right)\Phi_j+\left(-\frac{jr+\theta}{r}\right)\Phi_{j+1}, & \theta\in[-(j+1)r,-jr],\\ j=0,1,\dots,N-1.\end{cases} \tag{3.36}$$

It follows directly from Eq. (3.8) that the solution of the equation with the initial matrix function $\Phi(\theta)$, $\theta\in[-h,0]$, satisfies the equality

$$U(\tau,\Phi)\mathrm{e}^{-A_0\tau}=\Phi_0+\sum_{k=1}^{m}\int_0^{\tau}U(\xi-h_k,\Phi)A_k\mathrm{e}^{-A_0\xi}\mathrm{d}\xi.$$

For $\tau=jr$ we have

$$U(jr,\Phi)\mathrm{e}^{-A_0jr}=\Phi_0+\sum_{k=1}^{m}\int_0^{jr}U(\xi-h_k,\Phi)A_k\mathrm{e}^{-A_0\xi}\mathrm{d}\xi.$$

Comparing matrices $U(jr,\Phi)$ and $U((j+1)r,\Phi)$ we arrive at the equality

$$U((j+1)r,\Phi)\mathrm{e}^{-A_0r}-U(jr,\Phi)=\sum_{k=1}^{m}\int_{jr}^{(j+1)r}U(\xi-h_k,\Phi)A_k\mathrm{e}^{-A_0(\xi-jr)}\mathrm{d}\xi.$$

Let us express delays h_k as $h_k = j_k r + \theta_k,\ \theta_k \in [0, r)$, and introduce the matrices $U_k = U(kr, \Phi)$, $k = 0, 1, 2, \dots, N$. If we define the new integration variable by $\eta = \xi - \theta_k - jr$, then the previous equality can be written as

$$U_{j+1}\mathrm{e}^{-A_0 r} - U_j = \sum_{k=1}^{m} \int_{-\theta_k}^{r-\theta_k} U((j-j_k)r+\eta, \Phi)A_k \mathrm{e}^{-A_0(\eta+\theta_k)}\mathrm{d}\eta. \tag{3.37}$$

Each integral on the right-hand side of (3.37) can be written as follows:

$$\begin{aligned} L_k &= \int_{-\theta_k}^{r-\theta_k} U((j-j_k)\,r+\eta, \Phi)A_k \mathrm{e}^{-A_0\eta}\mathrm{d}\eta\,\mathrm{e}^{-A_0\theta_k} \\ &= \int_{-\theta_k}^{0} U((j-j_k)\,r+\eta, \Phi)A_k \mathrm{e}^{-A_0\eta}\mathrm{d}\eta\,\mathrm{e}^{-A_0\theta_k} \\ &\quad + \int_{0}^{r-\theta_k} U((j-j_k)\,r+\eta, \Phi)A_k \mathrm{e}^{-A_0\eta}\mathrm{d}\eta\,\mathrm{e}^{-A_0\theta_k}. \end{aligned} \tag{3.38}$$

To obtain an approximate value of the integral we replace the matrix $U(\tau, \Phi)$ under the integral by its piecewise linear approximation

$$\widehat{U}(\tau) = \begin{cases} \left(1 - \frac{\tau - jr}{r}\right)U_j + \left(\frac{\tau - jr}{r}\right)U_{j+1}, & \tau \in [jr, (j+1)r], \\ j = 0, 1, \dots, N-1. \end{cases}$$

The sign of the argument of $U(\cdot)$ under the integral (3.38) depends on the factor $(j - j_k)$. The following lemma provides evaluations of the integral for different values of the factor.

Lemma 3.10. *The piecewise linear approximation $\widehat{U}(\tau)$ provides the following expressions for integral (3.38).*

- *If $j - j_k \geq 1$, then the integral*

$$\begin{aligned} \widehat{L}_k &= \int_{-\theta_k}^{r-\theta_k} \widehat{U}((j-j_k)\,r+\eta)A_k \mathrm{e}^{-A_0(\eta+\theta_k)}\mathrm{d}\eta \\ &= U_{j-j_k-1}P_k + U_{j-j_k}Q_k + U_{j-_k+1}R_k; \end{aligned}$$

- *If* $j - j_k \le -1$, *then the integral*

$$\widehat{L}_k = \int_{-\theta_k}^{r-\theta_k} \widehat{U}((j-j_k)r+\eta)A_k \mathrm{e}^{-A_0(\eta+\theta_k)}\mathrm{d}\eta$$
$$= \Phi_{j_k-j+1}P_k + \Phi_{j_k-j}Q_k + \Phi_{j_k-j-1}R_k;$$

- *If* $j - j_k = 0$, *then the integral*

$$\widehat{L}_k = \int_{-\theta_k}^{r-\theta_k} \widehat{U}((j-j_k)r+\eta)A_k \mathrm{e}^{-A_0(\eta+\theta_k)}\mathrm{d}\eta$$
$$= \Phi_1 P_k + U_0 Q_k + U_1 R_k,$$

where the matrices

$$P_k = A_k \left[\int_{-\theta_k}^{0} \left(-\frac{\eta}{r}\right) \mathrm{e}^{-A_0\eta}\mathrm{d}\eta \right] \mathrm{e}^{-A_0\theta_k},$$

$$Q_k = A_k \left[\int_{-\theta_k}^{0} \left(1+\frac{\eta}{r}\right) \mathrm{e}^{-A_0\eta}\mathrm{d}\eta + \int_{0}^{r-\theta_k} \left(1-\frac{\eta}{r}\right) \mathrm{e}^{-A_0\eta}\mathrm{d}\eta \right] \mathrm{e}^{-A_0\theta_k},$$

$$R_k = A_k \left[\int_{0}^{r-\theta_k} \left(\frac{\eta}{r}\right) \mathrm{e}^{-A_0\eta}\mathrm{d}\eta \right] \mathrm{e}^{-A_0\theta_k}.$$

Thus, all the summands on the right-hand side of (3.37) can be expressed in the terms of the matrices Φ_k and U_j, and we arrive at the set of N linear matrix equations for these matrices. Application of the symmetry property at the partition points

$$U_j = \Phi_j^T, \quad j = 0,1,\ldots,N, \tag{3.39}$$

makes it possible to exclude the matrices U_j from these matrix equations and obtain a set of N matrix equations for $N+1$ matrices Φ_k, $k = 0,1,\ldots,N$. If we add to this set algebraic condition (3.31) expressed in the terms of the matrices

$$-W = A_0^T\Phi_0 + \Phi_0 A_0 + \sum_{k=1}^{m} A_k^T \left[\left(1-\frac{\theta_k}{r}\right)\Phi_{j_k}^T + \left(\frac{\theta_k}{r}\right)\Phi_{j_k+1}^T \right]$$
$$+ \sum_{k=1}^{m} \left[\left(1-\frac{\theta_k}{r}\right)\Phi_{j_k} + \left(\frac{\theta_k}{r}\right)\Phi_{j_k+1} \right] A_k,$$

then we finally obtain the system of $N+1$ matrix equations for the matrices.

The solution of the system provides the matrices Φ_k, $k=0,1,\dots,N$. Now formula (3.36) defines the desired approximation of the initial matrix function.

Approximate Lyapunov Matrices

Now with the piecewise linear initial matrix $\Phi(\theta)$, $\theta\in[-h,0]$, computed at the previous stage, we compute the corresponding solution of matrix equation (3.8). This can be done by the step-by-step method. The computed solution, $\widehat{U}(\tau,\Phi)$, defines the desired approximation of the Lyapunov matrix $U(\tau)$.

3.4.4 Error Estimation

In the case of general delays the semianalytic method is not applicable, so it is not possible to evaluate the quality of the approximate Lyapunov matrix by means of direct comparison with the exact Lyapunov matrix obtained by the semianalytic method. In this section we provide a different approach to the estimation of the quality of the approximate Lyapunov matrices.

By construction, the matrix $\widehat{U}(\tau)=\widehat{U}(\tau,\Phi)$ satisfies dynamic property (3.8), and $\widehat{U}(0)=\Phi_0$ is symmetric. But it is not required that properties (3.9) and (3.10) be satisfied. The error matrix

$$\Delta(\tau)=\widehat{U}(\tau)-\Phi^T(-\tau),\quad \tau\in[0,h],$$

describes the discrepancy of the symmetry property. The algebraic property (3.10) can be written in the form

$$\sum_{j=0}^{m}\left[U^T(h_j)A_j+A_j^TU(h_j)\right]=-W.$$

Let us define the matrix $\widehat{W}$ as follows:

$$\sum_{j=0}^{m}\left(\left[\widehat{U}(h_j)\right]^TA_j+A_j^T\widehat{U}(h_j)\right)=-\widehat{W}.$$

Then the matrix $\Delta W=W-\widehat{W}$ evaluates the violation of the algebraic property.

Given the positive-definite matrices W_j, $j=0,1,\dots,2m$, let system (3.1) satisfy the Lyapunov condition. Then there exists a Lyapunov matrix $U(\tau)$ associated with $W=W_0+\sum_{j=1}^{m}(W_j+h_jW_{m+j})$. The Lyapunov matrix defines the complete type functional (3.15). It is convenient to present the functional in the form

$$v(\varphi) = \varphi^T(0)U(0)\varphi(0) + \sum_{j=1}^{m} 2\varphi^T(0) \int_{-h_j}^{0} U^T(\theta+h_j)A_j\varphi(\theta)\mathrm{d}\theta$$

$$+ \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \left(\int_{-h_j}^{0} U^T(\theta_2+h_j-\theta_1-h_k)A_j\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$+ \sum_{j=1}^{m} \int_{-h_j}^{0} \varphi^T(\theta)\left[W_j+(h_j+\theta)W_{m+j}\right]\varphi(\theta)\mathrm{d}\theta.$$

Let us replace matrix $U(\tau)$ in the preceding functional by the approximate matrix $\widehat{U}(\tau)$ and denote the new functional by $\widehat{v}(\varphi)$:

$$\widehat{v}(\varphi) = \varphi^T(0)\widehat{U}(0)\varphi(0) + \sum_{j=1}^{m} 2\varphi^T(0) \int_{-h_j}^{0} \left[\widehat{U}(\theta+h_j)\right]^T A_j\varphi(\theta)\mathrm{d}\theta$$

$$+ \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \left(\int_{-h_j}^{0} \left[\widehat{U}(\theta_2+h_j-\theta_1-h_k)\right]^T A_j\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$+ \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta)\left[W_k+(h_k+\theta)W_{m+k}\right]\varphi(\theta)\mathrm{d}\theta.$$

Lemma 3.11. *The time derivative of $\widehat{v}(\varphi)$ along the solutions of system (3.1) is equal to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{v}(x_t) = -\widehat{w}(x_t)$$

$$= -w(x_t) + x^T(t)\left(W + \sum_{j=0}^{m}\left[\left(\widehat{U}(h_j)\right)^T A_j + A_j^T\widehat{U}(h_j) \right] \right) x(t)$$

$$+ \sum_{k=1}^{m}\sum_{j=1}^{m} x^T(t)A_j^T \int_{-h_k}^{0} \left(\widehat{U}(-\theta-h_k+h_j) \right.$$

$$\left. - \left[\widehat{U}(\theta+h_k-h_j)\right]^T \right) A_k x(t+\theta)\mathrm{d}\theta$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{m} x^T(t-h_j)A_j^T \int_{-h_k}^{0} \left(\left[\widehat{U}(\theta+h_k)\right]^T \right.$$

$$\left. - \widehat{U}(-\theta-h_k) \right) A_k x(t+\theta)\mathrm{d}\theta, \quad t \geq 0.$$

Proof. We first compute the time derivative of the individual terms of the functional. For the first term,

$$\widehat{R}_0(t) = x^T(t)\widehat{U}(0)x(t),$$

we have

$$\frac{\mathrm{d}\widehat{R}_0(t)}{\mathrm{d}t} = 2x^T(t)\widehat{U}(0)\left[\sum_{j=0}^{n} A_j x(t-h_j)\right].$$

The time derivative of the term

$$\widehat{R}_j(t) = 2x^T(t)\int_{-h_j}^{0}\left[\widehat{U}(\theta+h_j)\right]^T A_j x(t+\theta)\mathrm{d}\theta$$

is equal to

$$\begin{aligned}\frac{\mathrm{d}\widehat{R}_j(t)}{\mathrm{d}t} &= 2\left[\sum_{j=0}^{n} A_j x(t-h_j)\right]^T\int_{-h_j}^{0}\left[\widehat{U}(\theta+h_j)\right]^T A_j x(t+\theta)\mathrm{d}\theta \\ &\quad +2x^T(t)\left(\left[\widehat{U}(h_j)\right]^T A_j x(t) - \left[\widehat{U}(0)\right]^T A_j x(t-h_j)\right. \\ &\quad \left. -\int_{-h_j}^{0}\left[\frac{\mathrm{d}\widehat{U}(\tau)}{\mathrm{d}\tau}\right]^T_{\tau=\theta+h_j} A_j x(t+\theta)\mathrm{d}\theta\right).\end{aligned}$$

Now we compute the time derivative of the term

$$\widehat{R}_{kj}(t) = \int_{-h_k}^{0} x^T(t+\theta_1)A_k^T\left(\int_{-h_j}^{0}\left[\widehat{U}(\theta_2+h_j-\theta_1-h_k)\right]^T A_j x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1.$$

It is equal to the expression

$$\begin{aligned}\frac{\mathrm{d}\widehat{R}_{kj}(t)}{\mathrm{d}t} &= x^T(t)A_k^T\left(\int_{-h_j}^{0}\left[\widehat{U}(\theta+h_j-h_k)\right]^T A_j x(t+\theta)\mathrm{d}\theta\right) \\ &\quad -x^T(t-h_k)A_k^T\left(\int_{-h_j}^{0}\left[\widehat{U}(\theta+h_j)\right]^T A_j x(t+\theta)\mathrm{d}\theta\right)\end{aligned}$$

$$+ x^T(t)A_j^T\left(\int_{-h_k}^{0}\widehat{U}(-\theta+h_j-h_k)A_kx(t+\theta)\mathrm{d}\theta\right)$$

$$- x^T(t-h_j)A_j^T\left(\int_{-h_k}^{0}\widehat{U}(-\theta-h_k)A_kx(t+\theta)\mathrm{d}\theta\right).$$

The time derivative of the term

$$\widehat{J}_k(t) = \int_{-h_k}^{0} x^T(t+\theta)\left[W_k+(h_k+\theta)W_{m+k}\right]x(t+\theta)\mathrm{d}\theta$$

is of the form

$$\frac{\mathrm{d}\widehat{J}_k(t)}{\mathrm{d}t} = x^T(t)\left[W_k+h_kW_{m+k}\right]x(t) - x^T(t-h_k)W_kx(t-h_k)$$
$$- \int_{-h_k}^{0} x^T(t+\theta)W_{m+k}x(t+\theta)\mathrm{d}\theta.$$

Now, collecting the computed time derivatives we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{v}(x_t) = x^T(t)\left(\sum_{j=0}^{m}\left[\left(\widehat{U}(h_j)\right)^TA_j+A_j^T\widehat{U}(h_j)\right]\right)x(t)$$

$$+ \sum_{j=1}^{m}2x^T(t)\int_{-h_j}^{0}\left[-\frac{\mathrm{d}\widehat{U}(\tau)}{\mathrm{d}\tau}+\widehat{U}(\tau)A_0+\sum_{k=1}^{m}\widehat{U}(\tau-h_k)A_k\right]_{\tau=\theta+h_j}^{T}A_jx(t+\theta)\mathrm{d}\theta$$

$$+ \sum_{k=1}^{m}\sum_{j=1}^{m}x^T(t)\int_{-h_k}^{0}A_j^T\left(\widehat{U}(-\theta-h_k+h_j)-\left[\widehat{U}(\theta+h_k-h_j)\right]^T\right)A_kx(t+\theta)\mathrm{d}\theta$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{m}x^T(t-h_j)A_j^T\int_{-h_k}^{0}\left(\left[\widehat{U}(\theta+h_k)\right]^T-\widehat{U}(-\theta-h_k)\right)A_kx(t+\theta)\mathrm{d}\theta$$

$$+ X^T(t)\left[\sum_{k=1}^{m}(W_k+h_kW_{m+k})\right]x(t) - \sum_{k=1}^{m}x^T(t-h_k)W_kx(t-h_k)$$

$$- \sum_{k=1}^{m}\int_{-h_k}^{0}x^T(t+\theta)W_{m+k}x(t+\theta)\mathrm{d}\theta.$$

Since for $\theta \in [-h_j, 0]$, $\theta + h_j \geq 0$, then

$$2x^T(t) \int_{-h_j}^{0} \left[-\frac{d\widehat{U}(\tau)}{d\tau} + \widehat{U}(\tau)A_0 + \sum_{k=1}^{m} \widehat{U}(\tau - h_k)A_0 \right]^T_{\tau=\theta+h_j} A_j x(t+\theta)d\theta = 0, \quad j = 1, 2, \ldots, m,$$

and we arrive at the equality

$$\begin{aligned}\frac{d}{dt}\widehat{v}(x_t) = -w(x_t) + x^T(t)\left(W + \sum_{j=0}^{m}\left[\left(\widehat{U}(h_j)\right)^T A_j + A_j^T \widehat{U}(h_j) \right] \right) x(t) \\ + \sum_{k=1}^{m}\sum_{j=1}^{m} x^T(t) \int_{-h_k}^{0} A_j^T \Big(\widehat{U}(-\theta - h_k + h_j) \\ - \left[\widehat{U}(\theta + h_k - h_j) \right]^T \Big) A_k x(t+\theta)d\theta \\ + \sum_{j=1}^{m}\sum_{k=1}^{m} x^T(t - h_j)A_j^T \int_{-h_k}^{0} \Big(\left[\widehat{U}(\theta + h_k) \right]^T \\ -\widehat{U}(-\theta - h_k) \Big) A_k x(t+\theta)d\theta.\end{aligned}$$

For $\theta \in [-h_k, 0]$, $\theta + h_k \geq 0$, hence

$$\begin{aligned}G_k(\theta) &= \left[\widehat{U}(\theta + h_k)\right]^T - \widehat{U}(-\theta - h_k) = \left[\widehat{U}(\theta + h_k)\right]^T - \Phi(-\theta - h_k) \\ &= \Delta^T(\theta + h_k), \quad k = 1, 2, \ldots, m,\end{aligned}$$

and

$$\begin{aligned}F_{kj}(\theta) &= \widehat{U}(-\theta - h_k + h_j) - \left[\widehat{U}(\theta + h_k - h_j)\right]^T \\ &= \begin{cases} -\Delta^T(\theta + h_k - h_j), & \text{if } \theta + h_k - h_j \geq 0 \\ \Delta(-\theta - h_k + h_j), & \text{if } \theta + h_k - h_j < 0. \end{cases}\end{aligned}$$

We propose to evaluate the quality of the approximate matrix $\widehat{U}(\tau)$ comparing the time derivative of the functional $\widehat{v}(x_t)$ with that of the functional $v(x_t)$: the smaller the difference between the time derivatives, the better the approximation. To this end, we define the quantities

$$\rho_0 = \left\| W - \widehat{W} \right\|, \quad \rho_1 = \sup_{\tau \in [0,h]} \|\Delta(\tau)\|, \quad a_i = \|A_i\|, \; i = 1, \dots, m.$$

Observe that

$$J_1(t) = x^T(t) \left(W + \sum_{j=0}^{m} \left[\left(\widehat{U}(h_j) \right)^T A_j + A_j^T \widehat{U}(h_j) \right] \right) x(t)$$
$$= x^T(t) \left(W - \widehat{W} \right) x(t);$$

thus

$$|J_1(t)| \le \rho_0 \|x(t)\|^2 .$$

Now we estimate the term

$$J_2(t) = x^T(t-h_j) A_j^T \int_{-h_k}^{0} \left[\left(\widehat{U}(\theta + h_k) \right)^T - \widehat{U}(-\theta - h_k) \right] A_k x(t+\theta) \mathrm{d}\theta$$
$$= x^T(t-h_j) A_j^T \int_{-h_k}^{0} G_k(\theta) A_k x(t+\theta) \mathrm{d}\theta.$$

Here we have

$$|J_2(t)| \le \frac{\rho_1}{2} a_j a_k \left(h_k \left\| x(t-h_j) \right\|^2 + \int_{-h_k}^{0} \|x(t+\theta)\|^2 \mathrm{d}\theta \right).$$

In a similar way we obtain the estimation of the term

$$J_3(t) = x^T(t) A_j^T \int_{-h_k}^{0} \left[\widehat{U}(-\theta - h_k + h_j) - \left(\widehat{U}(\theta + h_k - h_j) \right)^T \right] A_k x(t+\theta) \mathrm{d}\theta$$
$$= x^T(t) A_j^T \int_{-h_k}^{0} F_{kj}(\theta) A_k x(t+\theta) \mathrm{d}\theta,$$

$$|J_3(t)| \le \frac{\rho_1}{2} a_j a_k \left(h_k \|x(t)\|^2 + \int_{-h_k}^{0} \|x(t+\theta)\|^2 \mathrm{d}\theta \right).$$

Finally, we arrive at the inequality

$$\left| \frac{\mathrm{d}\widehat{v}(x_t)}{\mathrm{d}t} - \frac{\mathrm{d}v(x_t)}{\mathrm{d}t} \right| \leq \sum_{k=0}^{m} \varepsilon_k \|x(t-h_k)\|^2 + \sum_{k=1}^{m} \varepsilon_{m+k} \int_{-h_k}^{0} \|x(t+\theta)\|^2 \mathrm{d}\theta, \qquad (3.40)$$

where

$$\varepsilon_0 = \rho_0 + \frac{\rho_1}{2} \left(\sum_{k=1}^{m} a_k \right) \left(\sum_{j=1}^{m} a_j h_j \right),$$

$$\varepsilon_k = \frac{\rho_1}{2} a_k \left(\sum_{j=1}^{m} a_j h_j \right), \quad \varepsilon_{m+k} = \rho_1 a_k \left(\sum_{j=1}^{m} a_j \right), \quad k = 1, 2, \ldots, m. \qquad \square$$

Remark 3.5. If $\lambda_{\min}(W_k) > \varepsilon_k$, $k = 0, 1, \ldots, m$, and $\lambda_{\min}(W_{m+j}) > \varepsilon_{m+j}$, $j = 1, \ldots, m$, then the time derivative of the functional $\widehat{v}(x_t)$ remains negative definite.

The quantity

$$\chi = \max_{l=0,1,\ldots,2m} \left\{ \frac{\varepsilon_l}{\lambda_{\min}(W_l)} \right\} \qquad (3.41)$$

is proposed as a qualitative measure of the approximation of a Lyapunov matrix: the smaller the measure, the better the approximation.

3.5 Exponential Estimates

In this section we show how one can use the complete type functionals to obtain exponential estimates for the solutions of system (3.1).

Theorem 3.10. *Assume we have two functionals $v, w : PC([-h,0], R^n) \to R$ such that the following conditions are satisfied:*

1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2$ *for some positive* α_1, α_2.
2. *There exists* $\sigma > 0$ *for which* $2\sigma v(\varphi) \leq w(\varphi)$.
3. *Along the solutions of system (3.1) the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) = -w(x_t), \quad t \geq 0.$$

Then the solutions of the system admit the exponential estimate

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \mathrm{e}^{-\sigma t} \|\varphi\|_h, \quad t \geq 0. \qquad (3.42)$$

Proof. Given any $\varphi \in PC([-h,0],R^n)$, conditions 1 and 3 imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t(\varphi))+2\sigma v(x_t(\varphi)) \leq 0, \quad t \geq 0.$$

Integrating this inequality we get

$$v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

Then condition 1 yields

$$\alpha_1 \|x(t,\varphi)\|^2 \leq v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t} \leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

Comparing the left- and right-hand sides of the preceding inequalities we arrive at the exponential estimate (3.42). □

We will now show that, conversely, if system (3.1) is exponentially stable, then a complete type functional (3.15) satisfies the conditions of Theorem 3.10.

Theorem 3.11. *If system (3.1) is exponentially stable and $W_0, W_1, \ldots, W_{2m}$ are positive-definite $n \times n$ matrices, then there exist positive constants $\alpha_1, \alpha_2, \sigma$ such that complete type functional (3.15) and functional (3.14) satisfy the conditions of Theorem 3.10.*

Proof. We have already seen that the exponential stability of system (3.1) implies that functionals (3.15) and (3.14) satisfy the last condition of Theorem 3.10. Lemmas 3.4 and 3.5 provide positive α_1, α_2 that satisfy the first condition of Theorem 3.10. It is evident that

$$\sum_{k=0}^{m} \lambda_{\min}(W_k) \|\varphi(-h_k)\|^2 + \sum_{j=1}^{m} \lambda_{\min}(W_{m+j}) \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq w(\varphi).$$

By Lemma 3.7, there exist positive constants δ_r, $r = 0, 1, \ldots, m$, such that

$$v(\varphi) \leq \delta_0 \|\varphi(0)\|^2 + \sum_{j=1}^{m} \delta_j \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

Comparing the last two inequalities we conclude that if $\sigma > 0$ satisfies the inequalities

$$2\sigma\delta_0 \leq \lambda_{\min}(W_0), \text{ and } 2\sigma\delta_j \leq \lambda_{\min}(W_{m+j}), \quad j = 1, 2, \ldots, m,$$

then the second condition of Theorem 3.10 is satisfied. This concludes the proof. □

Remark 3.6. The preceding proof shows that the first $(m+1)$ terms in (3.14) are needed to prove that the corresponding complete type functional (3.15) satisfies the left-hand-side inequality of the first condition of Theorem 3.10. The last m terms in (3.14), along with the first one, are used to derive the second inequality of the condition.

Remark 3.7. Clearly the exponential estimate obtained in Theorem 3.11 depends on the choice of positive-definite matrices W_j, $j=0,1,\dots,2m$. These matrices may serve as free parameters in an optimization of the estimate.

3.6 Robustness Bounds

3.6.1 Robust Stability Conditions: General Case

Assume that system (3.1) is exponentially stable, and consider a perturbed system of the form

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = (A_0+\Delta_0)\,y(t) + \sum_{k=1}^{m}(A_k+\Delta_k)\,y(t-h_k), \quad t\geq 0, \tag{3.43}$$

where Δ_k, $k=0,1,\dots,m$, are unknown but the norm bounded matrices

$$\|\Delta_k\| \leq \rho_k, \quad k=0,1,\dots,m. \tag{3.44}$$

We would like to estimate the bounds ρ_k for which perturbed system (3.43) remains exponentially stable for all possible perturbations Δ_k satisfying (3.44).

Lemma 3.12. *Given the positive-definite matrices W_j, $j=0,1,\dots,2m$, and functional (3.15), the time derivative of the functional along the solutions of perturbed system (3.43) is equal to*

$$\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) = -w(y_t) + 2\left[\Delta_0 y(t) + \sum_{k=1}^{m}\Delta_k y(t-h_k)\right]^T l(y_t), \quad t\geq 0,$$

where

$$l(y_t) = U(0)y(t) + \sum_{k=1}^{m}\int_{-h_k}^{0} U(-\theta-h_k)A_k y(t+\theta)\mathrm{d}\theta.$$

Proof. The proof of the lemma is similar to that of Lemma 2.14. □

Let us define the following constants:

$$\lambda_{\min} = \min_{0\le j\le 2m}\left\{\lambda_{\min}(W_j)\right\}, \quad a_j = \left\|A_j\right\|,\ j=1,\dots,m, \quad \nu = \max_{\tau\in[0,h]}\|U(\tau)\|.$$

It is not difficult to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) \le -w(y_t)\left[1-\frac{2\nu}{\lambda_{\min}}\|\rho\|\left(1+\sum_{k=1}^{m}h_k a_k^2\right)^{\frac{1}{2}}\right],$$

where $\|\rho\| = \left(\rho_0^2+\rho_1^2+\dots+\rho_m^2\right)^{\frac{1}{2}}$. And we arrive at the following statement.

Theorem 3.12. *Let system (3.1) be exponentially stable. Given the positive-definite matrices W_k, $k=0,1,\dots,2m$, perturbed system (3.43) remains exponentially stable for all possible perturbations satisfying (3.44) if*

$$\|\rho\| < \frac{\lambda_{\min}}{2\nu}\left(1+\sum_{k=1}^{m}h_k a_k^2\right)^{-\frac{1}{2}}.$$

3.6.2 Robust Stability Conditions: Scalar Case

Let us consider scalar equation (3.33). Assume that it is exponentially stable, and consider the scalar perturbed equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \sum_{j=0}^{m}(a_j+\Delta_j)\,y(t-j\mathfrak{h}), \quad t\ge 0. \tag{3.45}$$

Here unknown values Δ_j, $j=0,1,\dots,m$, are assumed to satisfy the inequalities

$$\left|\Delta_j\right| \le \rho_j, \quad j=0,1,\dots,m, \tag{3.46}$$

where ρ_j are nonnegative numbers. We are going to find bounds on ρ_j, $j=0,1,\dots,m$, such that the perturbed equation remains exponentially stable for all Δ_j, $j=0,1,\dots,m$, satisfying (3.46).

To derive such bounds, we apply functional (3.15) constructed for the nominal equation (3.33). The first time derivative of the functional along the solutions of Eq. (3.45) is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) &= -w(y_t)\\ &\quad +2\mu\left[\sum_{j=0}^{m}\Delta_j y(t-j\mathfrak{h})\right]\left[u(0)y(t)+\sum_{k=1}^{m}a_k\int_{-k\mathfrak{h}}^{0}u(k\mathfrak{h}+\theta)y(t+\theta)\mathrm{d}\theta\right]\\ &= -w(y_t)+2\mu\sum_{j=0}^{m}\Delta_j y(t-j\mathfrak{h})u(0)y(t)\\ &\quad +2\mu\sum_{j=0}^{m}\sum_{k=1}^{m}\Delta_j y(t-j\mathfrak{h})a_k\int_{-k\mathfrak{h}}^{0}u(k\mathfrak{h}+\theta)y(t+\theta)\mathrm{d}\theta,\end{aligned}$$

where $\mu=\mu_0+\sum_{j=1}^{m}(\mu_j+j\mathfrak{h}\mu_{m+j})$.

We estimate the term

$$J_j(t)=\left|2\Delta_j y(t-j\mathfrak{h})u(0)y(t)\right|\leq \rho_j u(0)\left[y^2(t)+y^2(t-j\mathfrak{h})\right].$$

Now consider the term

$$\begin{aligned}J_{jk}(t) &= \left|2\Delta_j y(t-j\mathfrak{h})a_k\int_{-k\mathfrak{h}}^{0}u(k\mathfrak{h}+\theta)y(t+\theta)\mathrm{d}\theta\right|\\ &\leq \rho_j\left|a_k\right|\int_{-k\mathfrak{h}}^{0}\left|u(k\mathfrak{h}+\theta)\right|[y^2(t-j\mathfrak{h})+y^2(t+\theta)]\mathrm{d}\theta\\ &= \rho_j\left|a_k\right|y^2(t-j\mathfrak{h})\int_{-k\mathfrak{h}}^{0}\left|u(k\mathfrak{h}+\theta)\right|\mathrm{d}\theta+\rho_j\left|a_k\right|\int_{-k\mathfrak{h}}^{0}\left|u(k\mathfrak{h}+\theta)\right|y^2(t+\theta)\mathrm{d}\theta.\end{aligned}$$

These estimations generate the following upper bound for the time derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) &\leq -\mu\left[\frac{\mu_0}{\mu}-\left(\rho_0+\sum_{j=0}^{m}\rho_j\right)u(0)-\rho_0\sum_{k=1}^{m}\left|a_k\right|\int_{-k\mathfrak{h}}^{0}\left|u(k\mathfrak{h}+\theta)\right|\mathrm{d}\theta\right]y^2(t)\\ &\quad -\sum_{k=1}^{m}\mu\left[\frac{\mu_k}{\mu}-\rho_k u(0)-\rho_k\sum_{j=1}^{m}\left|a_j\right|\int_{-j\mathfrak{h}}^{0}\left|u(j\mathfrak{h}+\theta)\right|\mathrm{d}\theta\right]y^2(t-k\mathfrak{h})\\ &\quad -\sum_{j=1}^{m}\mu\int_{-j\mathfrak{h}}^{0}\left[\frac{\mu_{m+j}}{\mu}-\left|a_j u(j\mathfrak{h}+\theta)\right|\sum_{k=0}^{m}\rho_k\right]y^2(t+\theta)\mathrm{d}\theta.\end{aligned}$$

The next theorem follows directly from the last inequality.

Theorem 3.13. *Let Eq. (3.33) be exponentially stable. Then system (3.45) remains stable for all perturbations satisfying (3.46) if the values ρ_j, $j = 0,1,\dots,m$, are such that the following inequalities hold:*

- $$\frac{\mu_0}{\mu} \geq \left(\rho_0 + \sum_{j=0}^{m} \rho_j\right) u(0) + \rho_0 \sum_{k=1}^{m} |a_k| \int_{-k\mathfrak{h}}^{0} |u(k\mathfrak{h}+\theta)|\,\mathrm{d}\theta;$$

- *For $k = 1,2,\dots,m$*

$$\frac{\mu_k}{\mu} \geq \rho_k u(0) + \rho_k \sum_{j=1}^{m} |a_j| \int_{-j\mathfrak{h}}^{0} |u(j\mathfrak{h}+\theta)|\,\mathrm{d}\theta;$$

- *For $j = 1,2,\dots m$*

$$\frac{\mu_{m+j}}{\mu} > \left|a_j u(j\mathfrak{h}+\theta)\right| \sum_{k=0}^{m} \rho_k, \quad \theta \in [-j\mathfrak{h}, 0].$$

Remark 3.8. If ρ_j, $j = 0,1,\dots,m$, satisfy the conditions of Theorem 3.13, then the trivial solution of Eq. (3.45) remains exponentially stable even if the perturbations Δ_j, $j = 0,1,\dots,m$, are time varying or depend on y_t. The only two assumptions needed are that they are continuous with respect to their arguments and satisfy (3.46) for all values of the arguments.

3.7 Applications

3.7.1 Critical Values

A system with one basic delay is considered, i.e., we study the case where $h_k = k\mathfrak{h}$, $k = 1,\dots,m$, so the delay system is now of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{k=0}^{m} A_k x(t-k\mathfrak{h}), \quad t \geq 0. \tag{3.47}$$

Here $\mathfrak{h} > 0$ is the basic delay and $A_k \in R^{n\times n}$, $k = 0,\dots,m$. The characteristic quasipolynomial of the system is written as

$$f(s) = \det\left(sI - \sum_{k=0}^{m} \mathrm{e}^{-\mathfrak{h}ks} A_k\right). \tag{3.48}$$

It was shown in Lemma 3.9 that for the computation of the Lyapunov matrix of the system one must find a special solution of the delay-free system of matrix Eqs. (3.29).

The spectrum of delay system (3.47) and that of delay-free system (3.29) are connected, as is explained in the following statement.

Theorem 3.14. *Let s_0 be an eigenvalue of system (3.47) such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of delay-free system (3.29). Furthermore, the spectrum of the delay-free system is symmetrical with respect to the imaginary axis of the complex plane.*

Proof. The characteristic matrix of system (3.47) is

$$G(s) = sI - \sum_{k=0}^{m} A_k \mathrm{e}^{-k\mathfrak{h}s}.$$

Since s_0 and $-s_0$ are eigenvalues of the systems, there exist nonzero vectors γ and μ such that

$$\gamma^T G(s_0) = 0, \qquad G^T(-s_0)\mu = 0. \tag{3.49}$$

A complex number s belongs to the spectrum of delay-free system (3.29) if and only if there exists a nontrivial set of $2m$ constant matrices $X_j^{(0)}$, $j = -m, \ldots, 0, \ldots, m-1$ of dimension $n \times n$ such that

$$\begin{cases} sX_j^{(0)} = \sum_{k=0}^{m} X_{j-k}^{(0)} A_k, & j = 0, 1, \ldots, m-1 \\ sX_j^{(0)} = -\sum_{k=0}^{m} A_k^T X_{j+k}^{(0)}, & j = -m, -m+1, \ldots, -1. \end{cases} \tag{3.50}$$

Multiplying the first equality in (3.49) on the left-hand side by $\mathrm{e}^{j\mathfrak{h}s_0}\mu$, $j = 0, \ldots, m-1$, and the second equality (3.49) on the right-hand side by $\mathrm{e}^{j\mathfrak{h}s_0}\gamma^T$, $j = -m, \ldots, -1$, we obtain

$$\begin{cases} s_0 \mathrm{e}^{j\mathfrak{h}s_0}\mu\gamma^T - \sum_{k=0}^{m} \mu\gamma^T \mathrm{e}^{(j-k)\mathfrak{h}s_0} A_k = 0_{n\times n}, & j = 0, \ldots, m-1 \\ -s_0 \mathrm{e}^{j\mathfrak{h}s_0}\mu\gamma^T - \sum_{k=0}^{m} A_k^T \mathrm{e}^{(j+k)\mathfrak{h}s_0}\mu\gamma^T = 0_{n\times n}, & j = -m, -m+1, \ldots, -1. \end{cases}$$

If we define the matrices $X_j^{(0)} = \mu\gamma^T \mathrm{e}^{j\mathfrak{h}s_0}$, $j = -m, \ldots, -1, 0, 1, \ldots, m-1$, then these equalities take the form

$$\begin{cases} s_0 X_j^{(0)} = \sum_{k=0}^{m} X_{j-k}^{(0)} A_k, & j = 0, 1, \ldots, m-1 \\ s_0 X_j^{(0)} = -\sum_{k=0}^{m} A_k^T X_{j+k}^{(0)}, & j = -m, -m+1, \ldots, -1. \end{cases}$$

Since the matrices $X_j^{(0)}$, $j = -m, \ldots, -1, 0, 1, \ldots, m-1$, are not trivial, s_0 is an eigenvalue of the delay-free system of matrix equations (3.29). The same is true for $-s_0$.

The fact that the spectrum of system (3.29) is symmetrical with respect to the imaginary axis follows directly from the observation that if for s there exists a nontrivial set of matrices $X_j^{(0)}$, $j = -m, \ldots, 0, \ldots, m-1$, satisfying (3.50), then, applying the transposition operation to the equalities in (3.50), one can check that the matrices $\widehat{X}_j^{(0)} = [X_{-j-1}^{(0)}]^T$, $j = -m, \ldots, 0, \ldots, m-1$, satisfy (3.50) for $-s$. □

The following observations are useful for the analysis of the roots of the system of matrix equations (3.29).

Remark 3.9. The characteristic polynomial $p(s)$ of system (3.29) is of degree $2mn^2$. Because of the symmetry of the system spectrum with respect to the imaginary axis, the polynomial can be written as $p_1(\lambda)$, where $\lambda = s^2$. A numerically important consequence of this fact is that the problem of determining the purely imaginary roots of $p(s)$ reduces to finding nonpositive real roots of the real polynomial $p_1(\lambda)$ of degree mn^2.

Remark 3.10. System (3.29) does not depend on the value of $\mathfrak{h}$. Thus the spectrum of this system does not depend on $\mathfrak{h}$ either. On the other hand, the spectrum of the original system (3.47) depends on $\mathfrak{h}$.

When the matrices of system (3.47) depend continuously on parameters or the value of the basic delay $\mathfrak{h}$ is not fixed, we can exploit Theorem 3.14 to compute the critical values of the parameters, i.e., the values for which system (3.47) admits eigenvalues on the imaginary axis of the complex plane. As the first step, one must find the set $\mathcal{K}$ of parameters for which the polynomial $p_1(\lambda)$ of Remark 3.9 has negative real roots. The knowledge of the roots allows one to define for every member of the set $\mathcal{K}$ the corresponding set $\mathcal{S}$ of candidate critical frequencies of system (3.47). Then for each member of the set $\mathcal{K}$ one must compute the values of quasipolynomial (3.48) at the points of the corresponding set of the candidate critical frequencies. A candidate critical frequency for which the quasipolynomial vanishes is a critical frequency, and the corresponding system parameters are critical parameter values.

Let us consider the case where the only free system parameter is the basic delay $\mathfrak{h}$. In this case all the coefficients of the polynomial $p_1(\lambda)$ are given real numbers. The set of negative real roots of this polynomial defines the set $\mathcal{S}$ of candidate critical frequencies of system (3.47). Clearly, when this set is empty and system (3.47) is stable (unstable) for the zero basic delay, the system is delay independent stable (unstable). If this set is not empty, one can substitute the elements of $\mathcal{S}$ into the characteristic quasipolynomial of system (3.47) and compute the critical values of $\mathfrak{h}$ for which the quasipolynomial vanishes. Using these critical values one can determine the stability and instability intervals of the basic delay by applying methods reported in the literature.

Finally, we illustrate how the results of this section can be employed in the stability analysis of time-delay systems with delays that are multiple of a basic delay.

3.7.2 The $\mathcal{H}_2$ Norm of a Transfer Matrix

In some applications it is important to compute the value of the $\mathcal{H}_2$ norm of the transfer matrix of a control system. In this section we consider an exponentially stable control system of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{k=0}^{m} \left[A_k x(t-h_k) + B_k u(t-h_k) \right], \quad t \geq 0, \tag{3.51}$$
$$y(t) = Cx(t-h).$$

The transfer matrix of the system is of the form

$$F(s) = \mathrm{e}^{-hs} C \left(sI - \sum_{k=0}^{m} \mathrm{e}^{-h_k s} A_k \right)^{-1} \left(\sum_{j=0}^{m} \mathrm{e}^{-h_j s} B_j \right) = \mathrm{e}^{-hs} CH(s)B(s),$$

where the matrix $H(s)$ is the Laplace image of the fundamental matrix $K(t)$ of control system (3.51),

$$H(s) = \int_0^{\infty} K(t)\mathrm{e}^{-st}\mathrm{d}t = \left(sI - \sum_{k=0}^{m} \mathrm{e}^{-h_k s} A_k \right)^{-1}.$$

The $\mathcal{H}_2$ norm of the transfer matrix is defined as follows:

$$\begin{aligned}
\|F\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathrm{Trace}\left\{ F^T(\xi) F(-\xi) \right\} \mathrm{d}\xi \\
&= \mathrm{Trace} \left\{ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} B^T(\xi) H^T(\xi) C^T C H(-\xi) B(-\xi) \mathrm{d}\xi \right\} \\
&= \sum_{p=0}^{m} \sum_{q=0}^{m} \mathrm{Trace} \left\{ B_p^T \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} H^T(\xi) C^T C H(-\xi) \mathrm{e}^{-(h_p - h_q)\xi} \mathrm{d}\xi \right] B_q \right\}.
\end{aligned}$$

Here $\mathrm{Trace}\{Q\}$ is the trace of a square $n \times n$ matrix Q,

$$\mathrm{Trace}\{Q\} = \sum_{j=1}^{n} q_{jj}.$$

Applying the frequency domain expression for Lyapunov matrices (Sect. 3.3.1) we obtain the equality

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)C^T CH(-\xi)\mathrm{e}^{-(h_p-h_q)\xi}\mathrm{d}\xi = U(h_p-h_q),$$

where the matrix $U(\tau)$ on the right-hand side of the preceding equality is the Lyapunov matrix $U(\tau)$ of system (3.51), with the trivial input $u(t)\equiv 0$ associated with matrix $W=C^TC$.

As a result we arrive at the following expression for the $\mathcal{H}_2$ norm of the transfer matrix:

$$\|F\|_{\mathcal{H}_2} = \left(\sum_{p=0}^{m}\sum_{q=0}^{m}\mathrm{Trace}\left\{B_p^T U(h_p-h_q)B_q\right\}\right)^{\frac{1}{2}}.$$

3.8 Notes and References

Lyapunov matrices and functionals for an exponentially stable system (3.1) are studied in [55]; see also [33, 39, 42, 59].

The piecewise linear approximation of Lyapunov matrices and error estimation are presented in [12]; see also [29]. Upper and lower bounds for quadratic Lyapunov functionals (Lemmas 3.4–3.17) are reported in [42] and [38]. These bounds are used to derive exponential estimates for the solutions of time-delay systems in [38] (Theorem 3.11); for the robustness analysis of an uncertain time-delay system in [42], see Theorem 3.12.

As was mentioned in Sect. 2.13, in [26] an explicit expression for Lyapunov matrices of a general time-delay system is obtained. The proof of Theorem 3.5 is an adaptation of the original proof of the theorem in [26] to the case of systems with multiple delays. A detailed analysis of Theorem 3.5 in the case of scalar time-delay equations can be found in [23]. The uniqueness statement of Theorem 3.6 is proven in [37]. The results presented in Sect. 3.4.1 are reported in [60].

Theorem 3.14 was obtained in [40]. The special case of the statement, namely, the case of eigenvalues of system (3.1) on an imaginary axis, was first reported in [53], where the idea of exploiting the fact that these eigenvalues are also roots of the characteristic polynomial of delay-free system (3.29) in the computation of critical delay values was proposed; see also [36] and references therein.

Application of Lyapunov matrices to the computation of the $\mathcal{H}_2$ norm of the transfer matrix of control system (3.51) presented in Sect. 3.7.2 is due to [29].

Chapter 4
Systems with Distributed Delay

In this chapter a linear retarded type system with distributed delays is studied. First, we introduce quadratic functionals and Lyapunov matrices for the system. Then we present the existence and uniqueness conditions for the matrices and provide some numerical schemes for the computation of the matrices. In the last part of the chapter functionals of the complete type are introduced, and some applications of the functionals are discussed.

4.1 System Description

We start with the following retarded type time-delay system:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta, \quad t \geq 0. \tag{4.1}$$

Here A_0 and A_1 are given real $n \times n$ matrices, delay $h > 0$, and $G(\theta)$ is a continuous matrix defined for $\theta \in [-h, 0]$.

4.2 Quadratic Functionals

Given a symmetric matrix W, we are looking for a quadratic functional

$$v_0 : PC([-h,0], R^n) \to R$$

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such that along the solutions of system (4.1) the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0. \tag{4.2}$$

Definition 4.1. The matrix $U(\tau)$ is said to be a Lyapunov matrix of system (4.1) associated with a symmetric matrix W if it satisfies the following properties:

1. Dynamic property:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) = U(\tau)A_0 + U(\tau-h)A_1 + \int_{-h}^{0} U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau \geq 0; \tag{4.3}$$

2. Symmetry property:

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0; \tag{4.4}$$

3. Algebraic property:

$$\begin{aligned} -W &= U(0)A_0 + U(-h)A_1 + \int_{-h}^{0} U(\theta)G(\theta)\mathrm{d}\theta + A_0^T U(0) \\ &\quad + A_1^T U(h) + \int_{-h}^{0} G^T(\theta)U(-\theta)\mathrm{d}\theta. \end{aligned} \tag{4.5}$$

Remark 4.1. The algebraic property can also be written as

$$U'(+0) - U'(-0) = -W. \tag{4.6}$$

For a given matrix $U(\tau)$ we define on $PC([-h,0],R^n)$ a functional of the form

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta \\ &\quad + \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left(\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\ &\quad + 2\varphi^T(0)\int_{-h}^{0} \left(\int_{-h}^{\theta} U(\xi-\theta)G(\xi)\mathrm{d}\xi \right) \varphi(\theta)\mathrm{d}\theta \end{aligned}$$

$$+ 2\int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left(\int_{-h}^{0} \left[\int_{-h}^{\theta_2} U(h+\theta_1-\theta_2+\xi)G(\xi)\mathrm{d}\xi \right] \varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$+ \int_{-h}^{0} \varphi^T(\theta_1) \left\{ \int_{-h}^{0} \left[\int_{-h}^{\theta_1} G^T(\xi_1) \left(\int_{-h}^{\theta_2} U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right) \mathrm{d}\xi_1 \right] \right.$$

$$\left. \times\, \varphi(\theta_2)\mathrm{d}\theta_2 \right\} \mathrm{d}\theta_1. \tag{4.7}$$

We can now prove the theorem.

Theorem 4.1. *Let $U(\tau)$ be a Lyapunov matrix of system (4.1) associated with W. Then the time derivative of functional (4.7) along the solutions of the system satisfies equality (4.2).*

Proof. Let $x(t)$, $t \geq 0$, be a solution of system (4.1); then

$$v_0(x_t) = x^T(t)U(0)x(t) + 2x^T(t)\int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)\mathrm{d}\theta$$

$$+ \int_{-h}^{0} x^T(t+\theta_1)A_1^T \left(\int_{-h}^{0} U(\theta_1-\theta_2)A_1x(t+\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$+ 2x^T(t)\int_{-h}^{0} \left[\int_{-h}^{\theta} U(\xi-\theta)G(\xi)\mathrm{d}\xi \right] x(t+\theta)\mathrm{d}\theta$$

$$+ 2\int_{-h}^{0} x^T(t+\theta_1)A_1^T \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right) \right.$$

$$\left. \times\, x(t+\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

$$+ \int_{-h}^{0} x^T(t+\theta_1) \left(\int_{-h}^{0} \left[\int_{-h}^{\theta_1} G^T(\xi_1) \left(\int_{-h}^{\theta_2} U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right) \mathrm{d}\xi_1 \right] \right.$$

$$\left. \times\, x(t+\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1.$$

At the first stage we compute the time derivative of each term of the functional.

For the first term, $R_0(t) = x^T(t)U(0)x(t)$, the time derivative is computed as

$$\frac{\mathrm{d}}{\mathrm{d}t}R_0(t) = \underline{2x^T(t)U(0)A_0x(t)} + \underline{2x^T(t)U(0)A_1x(t-h)}$$
$$\underline{+2x^T(t)U(0)\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta}.$$

The time derivative of the term

$$R_1(t) = 2x^T(t)\int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)\mathrm{d}\theta$$
$$= 2x^T(t)\int_{t-h}^{t} [U(h+s-t)]^T A_1x(s)\mathrm{d}s$$

is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}R_1(t) = \underbrace{2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]^T \int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s}$$
$$+\underline{2x^T(t)U(-h)A_1x(t)} - \underline{2x^T(t)U(0)A_1x(t-h)}$$
$$\underbrace{\underbrace{-2x^T(t)\int_{t-h}^{t}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\Big|_{\tau=h+s-t}\right]^T A_1x(s)\mathrm{d}s}}.$$

For the term

$$R_2(t) = \int_{-h}^{0} x^T(t+\theta_1)A_1^T\left(\int_{-h}^{0} U(\theta_1-\theta_2)A_1x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$
$$= \int_{t-h}^{t} x^T(s_1)A_1^T\left(\int_{t-h}^{t} U(s_1-s_2)A_1x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) = x^T(t)A_1^T\int_{t-h}^{t} U(t-s)A_1x(s)\mathrm{d}s$$
$$-x^T(t-h)A_1^T\int_{t-h}^{t} U(t-h-s)A_1x(s)\mathrm{d}s$$

$$
\begin{aligned}
&+\left(\int_{t-h}^{t} x^T(s)A_1^T U(s-t)\mathrm{d}s\right)A_1x(t) \\
&-\left(\int_{t-h}^{t} x^T(s)A_1^T U(s-t+h)\mathrm{d}s\right)A_1x(t-h) \\
=\;& \underbrace{\underbrace{2x^T(t)\int_{t-h}^{t}[U(s-t)A_1]^T A_1x(s)\mathrm{d}s}}\\
&\underbrace{-2x^T(t-h)A_1^T\int_{t-h}^{t}U(t-s-h)A_1x(s)\mathrm{d}s}.
\end{aligned}
$$

Now we consider the term

$$
\begin{aligned}
R_3(t) &= 2x^T(t)\int_{-h}^{0}\left[\int_{-h}^{\theta}U(\xi-\theta)G(\xi)\mathrm{d}\xi\right]x(t+\theta)\mathrm{d}\theta \\
&= 2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{s-t}U^T(-\xi+s-t)G(\xi)\mathrm{d}\xi\right]x(s)\mathrm{d}s.
\end{aligned}
$$

Its time derivative is given as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_3(t) =\;& \underline{\underline{2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]^T\int_{t-h}^{t}\left[\int_{-h}^{s-t}U(\xi-s+t)G(\xi)\mathrm{d}\xi\right]x(s)\mathrm{d}s}} \\
&+\underline{2x^T(t)\left[\int_{-h}^{0}U(\xi)G(\xi)\mathrm{d}\xi\right]x(t)}-\underline{2x^T(t)U(0)\int_{t-h}^{t}G(s-t)x(s)\mathrm{d}s} \\
&-2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{s-t}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\Big|_{\tau=-\xi+s-t}\right]^T G(\xi)\mathrm{d}\xi\right)x(s)\mathrm{d}s.
\end{aligned}
$$

The time derivative of the next term

$$R_4(t) = 2\int_{-h}^{0} x^T(t+\theta_1)A_1^T \left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2} U(h+\theta_1-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right) x(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$
$$= 2\int_{t-h}^{t} x^T(s_1)A_1^T \left[\int_{t-h}^{t}\left(\int_{-h}^{s_2-t} U(h+s_1-s_2+\xi)G(\xi)\mathrm{d}\xi\right) x(s_2)\mathrm{d}s_2\right]\mathrm{d}s_1$$

is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}R_4(t) = 2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{s-t} A_1^T U(h+t-s+\xi)G(\xi)\mathrm{d}\xi\right)x(s)\mathrm{d}s$$
$$\underline{\underline{-2\left[A_1x(t-h)\right]^T\int_{t-h}^{t}\left(\int_{-h}^{s-t} U(t-s+\xi)G(\xi)\mathrm{d}\xi\right)x(s)\mathrm{d}s}}$$
$$\underbrace{\underbrace{+2\left[\int_{t-h}^{t} x^T(s)\left(\int_{-h}^{0} A_1^T U(h+s-t+\xi)G(\xi)\mathrm{d}\xi\right)\mathrm{d}s\right]x(t)}}$$
$$\underbrace{-2\int_{t-h}^{t}\int_{t-h}^{t} x^T(s_1)A_1^T U(h+s_1-t)G(s_2-t)x(s_2)\mathrm{d}s_1\mathrm{d}s_2}.$$

And, finally, the time derivative of the last term,

$$R_5(t) = \int_{-h}^{0} x^T(t+\theta_1)\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_1} G^T(\xi_1)\left(\int_{-h}^{\theta_2} U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\right.$$
$$\left.\times\ x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$
$$= \int_{t-h}^{t} x^T(s_1)\left(\int_{t-h}^{t}\left[\int_{-h}^{s_1-t} G^T(\xi_1)\left(\int_{-h}^{s_2-t} U(s_1-s_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\right.$$
$$\left.\times\ x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1,$$

can be computed as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_5(t) &= x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{0}G^T(\xi_1)\left(\int_{-h}^{s-t}U(t-s-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]x(s)\mathrm{d}s \\
&\quad+\left(\int_{t-h}^{t}x^T(s)\left[\int_{-h}^{s-t}G^T(\xi_1)\left(\int_{-h}^{0}U(s_1-t-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\mathrm{d}s\right)x(t) \\
&\quad-\int_{t-h}^{t}x^T(s_1)\left(\int_{t-h}^{t}\left[\int_{-h}^{s_2-t}G^T(s_1-t)U(-s_2+t+\xi)G(\xi)\mathrm{d}\xi\right]x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 \\
&\quad-\int_{t-h}^{t}x^T(s_1)\left(\int_{t-h}^{t}\left[\int_{-h}^{s_1-t}G^T(\xi)U(s_1-\xi-t)G(s_2-t)\mathrm{d}\xi\right]x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 \\
&= 2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{0}G^T(\xi_1)\left(\int_{-h}^{s-t}U(t-s-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]x(s)\mathrm{d}s \\
&\quad-2\underline{\underline{\left[\int_{t-h}^{t}G(s_1-t)x(s_1)\mathrm{d}s_1\right]^T\left[\int_{t-h}^{t}\left(\int_{-h}^{s_2-t}U(-s_2+t+\xi)G(\xi)\mathrm{d}\xi\right)x(s_2)\mathrm{d}s_2\right]}}.
\end{aligned}
$$

At the next stage we collect terms in the computed time derivatives. We start with the terms that are underlined by a single straight line. Their sum is

$$
\begin{aligned}
S_1(t) &= 2x^T(t)U(0)A_0x(t)+2x^T(t)U(-h)A_1x(t)+2x^T(t)\left[\int_{-h}^{0}U(\xi)G(\xi)\mathrm{d}\xi\right]x(t) \\
&= x^T(t)\left(U(0)A_0+U(-h)A_1+\int_{-h}^{0}U(\xi)G(\xi)\mathrm{d}\xi\right. \\
&\qquad\left.+A_0^TU(0)+A_1^TU^T(-h)+\int_{-h}^{0}G^T(\xi)U^T(\xi)\mathrm{d}\xi\right)x(t) \\
&= -x^T(t)Wx(t).
\end{aligned}
$$

Now we collect the terms underlined by a single curved line. Their sum is

$$S_2(t) = 2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} - A_1x(t-h) - \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T \times \int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s$$

$$= 2x^T(t)A_0^T \int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s. \tag{4.8}$$

The sum of the terms underlined by a double curved line is equal to

$$S_3(t) = 2x^T(t) \int_{t-h}^{t} \left[-\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) + U(\tau-h)A_1 \right.$$

$$\left. + \int_{-h}^{0} U(\tau+\xi)G(\xi)\mathrm{d}\xi \Bigg|_{\tau=h+s-t} \right]^T A_1x(s)\mathrm{d}s$$

$$= -2x^T(t)A_0^T \int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s,$$

and it is cancelled by (4.8). The sum of the terms underlined by a double straight line is equal to

$$S_4(t) = 2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} - A_1x(t-h) - \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T$$

$$\times \int_{t-h}^{t} \left(\int_{-h}^{s-t} U(\xi-s+t)G(\xi)\mathrm{d}\xi\right) x(s)\mathrm{d}s$$

$$= 2x^T(t)A_0^T \int_{t-h}^{t} \left(\int_{-h}^{s-t} U(\xi-s+t)G(\xi)\mathrm{d}\xi\right) x(s)\mathrm{d}s. \tag{4.9}$$

Finally, the sum of the nonunderlined terms is

$$S_5(t) = 2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{s-t}\left(-\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)+U(\tau-h)A_1\right.\right.$$
$$\left.\left.+\int_{-h}^{0}U(\tau+\xi_2)G(\xi_2)\mathrm{d}\xi_2\Bigg|_{\tau=-\xi+s-t}\right)^T G(\xi)\mathrm{d}\xi\right]x(s)\mathrm{d}s$$
$$= -2x^T(t)A_0^T\int_{t-h}^{t}\left(\int_{-h}^{s-t}U(t-s+\xi)G(\xi)\mathrm{d}\xi\right)x(s)\mathrm{d}s,$$

and it is cancelled by (4.9).

Summarizing our computations we arrive at the conclusion that the time derivative of the functional $v_0(\varphi)$ along the solutions of system (4.1) satisfies equality (4.2). □

4.3 Lyapunov Matrices: Existence Issue

In this section we study the existence issue for the Lyapunov matrices of system (4.1).

The characteristic function of the system is of the form

$$f(s) = \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1 - \int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right). \tag{4.10}$$

We define the matrix

$$H(s) = \left(sI - A_0 - \mathrm{e}^{-sh}A_1 - \int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right)^{-1}.$$

The poles of $H(s)$ form the spectrum,

$$\Lambda = \{\, s \mid f(s) = 0\,\},$$

of the system. If system (4.1) satisfies the Lyapunov condition, then the spectrum can be divided into two parts; the first one, $\Lambda^{(+)}$, includes eigenvalues with positive real part, whereas the second one, $\Lambda^{(-)}$, includes eigenvalues with negative real part.

Theorem 4.2 ([26]). *Let system (4.1) satisfy the Lyapunov condition; then for any symmetric matrix W matrix*

$$\widetilde{U}(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{H^T(s)WH(-s)\mathrm{e}^{-\tau s}, s_0\right\}$$
$$+ \sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{H^T(-s)WH(s)\mathrm{e}^{\tau s}, s_0\right\} \tag{4.11}$$

is a Lyapunov matrix of the system associated with W.

Proof. System (4.1) satisfies the Lyapunov condition, so neither the matrix $H(s)$ nor the matrix $H(-s)$ has a pole on the imaginary axis of the complex plane. Let ξ be a real number; then for sufficiently large $|\xi|$ the matrix $H^T(i\xi)WH(-i\xi)\mathrm{e}^{-i\tau\xi}$ is of the order $|\xi|^{-2}$. This means that the improper integral on the right-hand side of (4.11) is well defined for all real τ.

Part 1: The proof of symmetry property (4.4) coincides with that of Theorem 3.5.

Part 2: We address now the algebraic property. To check (4.5), we compute the following matrix:

$$\mathcal{O} = \widetilde{U}(0)A_0 + \widetilde{U}(-h)A_1 + \int_{-h}^{0}\widetilde{U}(\theta)G(\theta)\mathrm{d}\theta + A_0^T\widetilde{U}(0)$$
$$+ A_1^T\widetilde{U}(h) + \int_{-h}^{0} G^T(\theta)\widetilde{U}(-\theta)\mathrm{d}\theta$$
$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left(H^T(\xi)WH(-\xi)\left[A_0 + \mathrm{e}^{\xi h}A_1 + \int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\right.$$
$$\left. + \left[A_0 + \mathrm{e}^{-\xi h}A_1 + \int_{-h}^{0}\mathrm{e}^{\xi\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(\xi)WH(-\xi)\right)\mathrm{d}\xi$$
$$+ \sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{H^T(s)WH(-s)\left[A_0 + \mathrm{e}^{sh}A_1 + \int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right], s_0\right\}$$
$$+ \sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{H^T(-s)WH(s)\left[A_0 + \mathrm{e}^{-sh}A_1 + \int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right], s_0\right\}$$

$$+\sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{\left[A_0+\mathrm{e}^{-sh}A_1+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(s)WH(-s),s_0\right\}$$

$$+\sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{\left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(-s)WH(s),s_0\right\}.$$

It is a matter of simple calculation to verify the identities

$$H(s)\left[A_0+\mathrm{e}^{-sh}+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]=sH(s)-I$$

and

$$H(-s)\left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]=-sH(-s)-I.$$

Additionally,

$$\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} WH(-\xi)\mathrm{d}\xi=\langle\lambda=-\xi\rangle=\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} WH(\lambda)\mathrm{d}\lambda.$$

Now, the matrix $\mathcal{O}$ can be written as

$$\begin{aligned}\mathcal{O}=&-\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W+WH(\xi)\right]\mathrm{d}\xi\\&-\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W,s_0\right\}-\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W,s_0\right\}\\&-\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)W,s_0\right\}-\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)W,s_0\right\}.\end{aligned}$$

Since the Lyapunov condition implies that no poles of the matrix $H(-s)$ lie in the set $\Lambda^{(+)}$, the last two sums on the right-hand side of the preceding equality disappear and

$$\mathcal{O}=-\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W+WH(\xi)\right]\mathrm{d}\xi-\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W+WH(s),s_0\right\}.$$

The remainder of the proof of this part is identical to that of Theorem 3.5.

Part 3: Let us address property (4.3). For a given $\tau > 0$ we compute the matrix

$$F(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{U}(\tau) - \widetilde{U}(\tau)A_0 - \widetilde{U}(\tau-h)A_1 - \int_{-h}^{0}\widetilde{U}(\tau+\theta)G(\theta)\mathrm{d}\theta$$

$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\left[-\xi I - A_0 - \mathrm{e}^{\xi h}A_1 - \int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{-\tau\xi}\mathrm{d}\xi$$

$$+ \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)WH(-s)\left[-sI - A_0 - \mathrm{e}^{sh}A_1 - \int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{-\tau s}, s_0\right\}$$

$$+ \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)WH(s)\left[sI - A_0 - \mathrm{e}^{-sh}A_1 - \int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{\tau s}, s_0\right\}$$

$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)W\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W\mathrm{e}^{-\tau s}, s_0\right\}$$

$$+ \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)W\mathrm{e}^{\tau s}, s_0\right\}.$$

Since the matrix $H(-s)$ has no poles in the set $\Lambda^{(+)}$, the sum

$$\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)W\mathrm{e}^{\tau s}, s_0\right\} = 0_{n\times n},$$

and we obtain

$$F(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)W\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)W\mathrm{e}^{-\tau s}, s_0\right\}.$$

The remainder of the proof of this part repeats that of Theorem 3.5. □

Corollary 4.1. *If system (4.1) is exponentially stable, then the Lyapunov matrix associated with a symmetric matrix W can be written as*

$$U(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\mathrm{e}^{-\tau\xi}\mathrm{d}\xi$$

$$= \int_{0}^{\infty} K(t)WK(t+\tau)\mathrm{d}\tau.$$

Here $K(t)$ is the fundamental matrix of the system.

4.4 Lyapunov Matrices: Uniqueness Issue

Here we study the uniqueness issue for Lyapunov matrices.

Lemma 4.1. *Given an integral-differential system of the form*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}z(\tau) = Az(\tau) + \int_0^{\tau} B(s,\tau)z(s)\mathrm{d}s \quad \tau \geq 0, \tag{4.12}$$

where A is a constant matrix and $B(s,\tau)$ is a continuous bivariate matrix, the only solution of the system that satisfies the condition $z(0) = 0$ is the trivial one.

Proof. Given $H > 0$, let us consider the system on the segment $[0,H]$. Compute the values

$$a = \|A\|, \quad b = \max_{(s,\tau)\in[0,H]^2} \|B(s,\tau)\|.$$

Integrating Eq. (4.12) from 0 to τ we obtain

$$z(\tau) = z(0) + A\int_0^{\tau} z(\xi)\mathrm{d}\xi + \int_0^{\tau}\left(\int_0^{\xi} B(s,\xi)z(s)\mathrm{d}s\right)\mathrm{d}\xi.$$

Thus,

$$\begin{aligned}\|z(\tau)\| &\leq \|z(0)\| + a\int_0^{\tau}\|z(\xi)\|\,\mathrm{d}\xi + b\int_0^{\tau}\left(\int_0^{\xi}\|z(s)\|\,\mathrm{d}s\right)\mathrm{d}\xi \\ &= \|z(0)\| + a\int_0^{\tau}\|z(\xi)\|\,\mathrm{d}\xi + b\int_0^{\tau}(\tau - s)\,\|z(s)\|\,\mathrm{d}s \\ &\leq \|z(0)\| + (a + bH)\int_0^{\tau}\|z(s)\|\,\mathrm{d}s.\end{aligned}$$

Now, by the Gronwall lemma,

$$\|z(\tau)\| \leq \mathrm{e}^{(a+bH)\tau}\,\|z(0)\|, \quad \tau \in [0,H].$$

In our case $z(0) = 0$, and we arrive at the conclusion that

$$z(\tau) = 0, \quad \tau \in [0,H].$$

□

Theorem 4.3. *Let system (4.1) satisfy the Lyapunov condition. Then for any symmetric matrix W there exists a unique Lyapunov matrix associated with W.*

Proof. Part 1: The fact that under the theorem condition matrix (4.11) satisfies Definition 4.1 was demonstrated in Theorem 4.2. Assume that for a given symmetric matrix W there exist two Lyapunov matrices, $U^{(1)}(\tau)$ and $U^{(2)}(\tau)$. Each of the matrices defines the corresponding functional, $v_0^{(j)}(\varphi)$, $j=1,2$, of the form (4.7). The functionals satisfy the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0^{(j)}(x_t) = -x^T(t)Wx(t), \quad j=1,2,$$

along the solutions of system (4.1). The difference, $\Delta v(x_t) = v_0^{(2)}(x_t) - v_0^{(1)}(x_t)$, is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(x_t) = 0, \quad t \geq 0,$$

and we obtain that for any $\varphi \in PC([-h,0],R^n)$ the identity

$$\Delta v(x_t(\varphi)) = \Delta v(\varphi), \quad t \geq 0, \tag{4.13}$$

holds along the solution $x(t,\varphi)$ of the system. In the case where system (4.1) is exponentially stable, $x_t(\varphi) \to 0_h$ as $t \to \infty$, and we arrive at the conclusion that

$$\Delta v(\varphi) = 0, \quad \varphi \in PC([-h,0],R^n). \tag{4.14}$$

If system (4.1) is not exponentially stable, then by the Lyapunov condition it has no eigenvalues on the imaginary axis of the complex plane, and there is a finite number of the eigenvalues in the open right half-plane of the complex plane. Let $\chi > 0$ be an upper bound for the real part of the eigenvalues in the right half-plane. Only a finite number of the system eigenvalues, $s_1, s_2, \ldots, s_N$, lies in the vertical stripe

$$Z = \{\ s \mid -\chi \leq \mathrm{Re(s)} \leq \chi\ \}$$

of the complex plane. Every solution $x(t,\varphi)$ of the system can be presented as the sum

$$x(t,\varphi) = x^{(1)}(t) + x^{(2)}(t),$$

where $x^{(1)}(t)$ corresponds to the part of the system spectrum that lies in Z and $x^{(2)}(t)$ corresponds to the rest of the spectrum, which lies to the left of the vertical line $\mathrm{Re(s)} = -\chi$.

The first term, $x^{(1)}(t)$, is a finite sum of the form

$$x^{(1)}(t) = \sum_{\ell=1}^{N} \mathrm{e}^{s_\ell t} p^{(\ell)}(t),$$

where $p^{(\ell)}(t)$ is a polynomial with vector coefficients of degree less than the multiplicity of s_ℓ as a zero of the system characteristic function (4.10), $\ell = 1, 2, \ldots, N$.

The second term, $x^{(2)}(t)$, admits an upper estimate of the form

$$\left\| x^{(2)}(t) \right\| \leq c\mathrm{e}^{-(\chi+\varepsilon)t}, \quad t \geq 0. \tag{4.15}$$

Here c is a positive constant and ε is a small positive number.

The functional $\Delta v(x_t(\varphi))$ can be decomposed as follows:

$$\Delta v\left(x_t(\varphi)\right) = \Delta v\left(x_t^{(1)}\right) + 2\Delta z\left(x_t^{(1)}, x_t^{(2)}\right) + \Delta v\left(x_t^{(2)}\right),$$

where

$$\begin{aligned}
\Delta z\left(x_t^{(1)}, x_t^{(2)}\right) &= \left[x^{(1)}(t)\right]^T \Delta U(0) x^{(2)}(t) \\
&+ \left[x^{(1)}(t)\right]^T \int_{-h}^{0} \Delta U(-h-\theta) A_1 x^{(2)}(t+\theta) \mathrm{d}\theta \\
&+ \left[x^{(2)}(t)\right]^T \int_{-h}^{0} \Delta U(-h-\theta) A_1 x^{(1)}(t+\theta) \mathrm{d}\theta \\
&+ \left[x^{(1)}(t)\right]^T \int_{-h}^{0} \left[\int_{-h}^{\theta} \Delta U(\xi-\theta) G(\xi) \mathrm{d}\xi\right] x^{(2)}(t+\theta) \mathrm{d}\theta \\
&+ \left[x^{(2)}(t)\right]^T \int_{-h}^{0} \left[\int_{-h}^{\theta} \Delta U(\xi-\theta) G(\xi) \mathrm{d}\xi\right] x^{(1)}(t+\theta) \mathrm{d}\theta \\
&+ \int_{-h}^{0} \left[x^{(1)}(t+\theta_1)\right]^T A_1^T \left(\int_{-h}^{0} \Delta U(\theta_1-\theta_2) A_1 x^{(2)}(t+\theta_2) \mathrm{d}\theta_2\right) \mathrm{d}\theta_1 \\
&+ \int_{-h}^{0} \left[x^{(1)}(t+\theta_1)\right]^T A_1^T \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} \Delta U(h+\theta_1-\theta_2+\xi_2) G(\xi_2) \mathrm{d}\xi_2\right)\right. \\
&\left. \times x^{(2)}(t+\theta_2) \mathrm{d}\theta_2\right] \mathrm{d}\theta_1 \\
&+ \int_{-h}^{0} \left[x^{(2)}(t+\theta_1)\right]^T A_1^T \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} \Delta U(h+\theta_1-\theta_2+\xi_2) G(\xi_2) \mathrm{d}\xi_2\right)\right.
\end{aligned}$$

$$\times x^{(1)}(t+\theta_2)\mathrm{d}\theta_2\Bigg]\mathrm{d}\theta_1$$

$$+\int_{-h}^{0}\left[x^{(1)}(t+\theta_1)\right]^T\left\{\int_{-h}^{0}\left(\int_{-h}^{\theta_1}G^T(\xi_1)\right.\right.$$

$$\left.\left.\times\left[\int_{-h}^{\theta_2}\Delta U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right]\mathrm{d}\xi_1\right)x^{(2)}(t+\theta_2)\mathrm{d}\theta_2\right\}\mathrm{d}\theta_1.$$

On the one hand, since $x^{(1)}(t)$ and $x^{(2)}(t)$ are solutions of system (4.1), $\Delta v\left(x_t^{(1)}\right)$ and $\Delta v\left(x_t^{(2)}\right)$ maintain constant values, and we conclude that $\Delta z\left(x_t^{(1)},x_t^{(2)}\right)$ is also constant. On the other hand, the choice of χ and inequality (4.15) guarantee that

$$\Delta v\left(x_t^{(2)}\right)\to 0,\text{ and }\Delta z\left(x_t^{(1)},x_t^{(2)}\right)\to 0,\quad \text{as } t\to\infty.$$

This means that

$$\Delta v\left(x_t^{(2)}\right)=0,\text{ and }\Delta z\left(x_t^{(1)},x_t^{(2)}\right)=0,\quad t\geq 0.$$

The first summand, $\Delta v\left(x_t^{(1)}\right)$, can be written as follows:

$$\Delta v\left(x_t^{(1)}\right)=\sum_{\ell=1}^{N}\sum_{r=1}^{N}\mathrm{e}^{(s_\ell+s_r)t}\alpha_{\ell r}(t),$$

where the functions $\alpha_{\ell r}(t)$, $\ell,r=1,2,\ldots,N$, are of the form

$$\alpha_{\ell r}(t)=\left[p^{(\ell)}(t)\right]^T\Delta U(0)p^{(r)}(t)+2\left[p^{(\ell)}(t)\right]^T\int_{-h}^{0}\Delta U(-h-\theta)A_1\mathrm{e}^{s_r\theta}p^{(r)}(t+\theta)\mathrm{d}\theta$$

$$+2\left[p^{(\ell)}(t)\right]^T\int_{-h}^{0}\left[\int_{-h}^{\theta}\Delta U(\xi-\theta)G(\xi)\mathrm{d}\xi\right]\mathrm{e}^{s_r\theta_2}p^{(r)}(t+\theta)\mathrm{d}\theta$$

$$+\int_{-h}^{0}\left[\mathrm{e}^{s_\ell\theta_1}p^{(\ell)}(t+\theta_1)\right]^T A_1^T\left(\int_{-h}^{0}\Delta U(\theta_1-\theta_2)A_1\mathrm{e}^{s_r\theta_2}p^{(r)}(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$

$$+2\int_{-h}^{0}\left[\mathrm{e}^{s_\ell\theta_1}p^{(\ell)}(t+\theta_1)\right]^T A_1^T$$

$$\times \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} \Delta U(h+\theta_1-\theta_2+\xi_2) G(\xi_2) \mathrm{d}\xi_2 \right) \mathrm{e}^{s_r\theta_2} p^{(r)}(t+\theta_2) \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

$$+ \int_{-h}^{0} \left[\mathrm{e}^{s_\ell\theta_1} p^{(\ell)}(t+\theta_1) \right]^T$$

$$\times \left\{ \int_{-h}^{0} \left(\int_{-h}^{\theta_1} G^T(\xi_1) \left[\int_{-h}^{\theta_2} \Delta U(\theta_1-\theta_2-\xi_1+\xi_2) G(\xi_2) \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \right) \right.$$

$$\left. \times\, \mathrm{e}^{s_r\theta_2} p^{(r)}(t+\theta_2) \mathrm{d}\theta_2 \right\} \mathrm{d}\theta_1.$$

A careful inspection of $\alpha_{\ell r}(t)$ reveals that it is a polynomial in t of degree less than the sum of the multiplicities of s_ℓ and s_r as zeros of the characteristic function (4.10). This means that identity (4.13) takes the form

$$\sum_{\ell=1}^{N} \sum_{r=1}^{N} \mathrm{e}^{(s_\ell+s_r)t} \alpha_{\ell r}(t) = \mathrm{e}^{0t} \Delta v(\varphi), \quad t \geq 0.$$

Part 2: According to the Lyapunov condition, no one of the sums $(s_\ell + s_r)$, $\ell, r \in \{1,2,\ldots,N\}$, is equal to zero. Therefore, by Lemma 3.8, we conclude from the last identity that equality (4.14) holds for any initial function $\varphi \in PC([-h,0],R^n)$.

Part 3: Equality (4.14) can be written as follows:

$$\begin{aligned} 0 = {} & \varphi^T(0)\Delta U(0)\varphi(0) \\ & + 2\varphi^T(0) \int_{-h}^{0} \left[\Delta U(-h-\theta)A_1 + \int_{-h}^{\theta} \Delta U(\xi-\theta)G(\xi)\mathrm{d}\xi \right] \varphi(\theta)\mathrm{d}\theta \\ & + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1) \left[A_1^T \Delta U(\theta_1-\theta_2)A_1 \right. \\ & + 2\int_{-h}^{\theta_2} A_1^T \Delta U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \\ & \left. + \int_{-h}^{\theta_1}\int_{-h}^{\theta_2} G^T(\xi_1)\Delta U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_1 \right] \varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1. \end{aligned} \tag{4.16}$$

For a given vector $\gamma \in R^n$ we define the initial function

$$\varphi^{(1)}(\theta) = \begin{cases} \gamma, & \text{for } \theta = 0 \\ 0, & \text{for } \theta \in [-h, 0) \end{cases}.$$

For this function equality (4.16) takes the form

$$\gamma^T \Delta U(0) \gamma = 0.$$

Since the last equality holds for any vector γ and the matrix $\Delta U(0)$ is symmetric, we conclude that

$$\Delta U(0) = 0_{n\times n}. \tag{4.17}$$

Now, given vectors $\gamma \in R^n$ and $\mu \in R^n$, let us select $\tau \in (0, h]$ and $\varepsilon > 0$ such that $-\tau + \varepsilon < 0$. Then we define the following initial function:

$$\varphi^{(2)}(\theta) = \begin{cases} \gamma, & \text{for } \theta = 0, \\ \mu, & \text{for } \theta \in [-\tau, -\tau + \varepsilon], \\ 0, & \text{for all other points of segment } [-h, 0]. \end{cases}$$

For this initial function equality (4.16) takes the form

$$0 = 2\varepsilon \gamma^T \left[\Delta U(\tau - h) A_1 + \int_{-h}^{-\tau} \Delta U(\tau + \xi) G(\xi) \mathrm{d}\xi \right] \mu + o(\varepsilon),$$

where

$$\lim_{\varepsilon \to +0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Since γ and μ are arbitrary vectors and $\varepsilon > 0$ may be arbitrarily small, we conclude that the equality

$$\Delta U(\tau - h) A_1 + \int_{-h}^{-\tau} \Delta U(\tau + \xi) G(\xi) \mathrm{d}\xi = 0_{n\times n}$$

holds for $\tau \in (0, h]$. By continuity arguments, we obtain

$$\Delta U(\tau - h) A_1 + \int_{-h}^{-\tau} \Delta U(\tau + \xi) G(\xi) \mathrm{d}\xi = 0_{n\times n}, \quad \tau \in [0, h]. \tag{4.18}$$

The matrix $\Delta U(\tau)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau)=\Delta U(\tau)A_0+\Delta U(\tau-h)A_1+\int_{-h}^{0}\Delta U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau\in[0,h].$$

Condition (4.18) makes it possible to present the preceding equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau)=\Delta U(\tau)A_0+\int_{-\tau}^{0}\Delta U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau\in[0,h]$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau)=\Delta U(\tau)A_0+\int_{0}^{\tau}\Delta U(s)G(s-\tau)\mathrm{d}s, \quad \tau\in[0,h].$$

We are looking for a solution of this equation that satisfies condition (4.17). By Lemma 4.1, the solution is trivial, and

$$\Delta U(\tau)=U^{(2)}(\tau)-U^{(1)}(\tau)=0_{n\times n}, \quad \tau\in[0,h]. \qquad \square$$

4.5 Lyapunov Matrices: Computational Issue

In this section we present some approaches to the computation of Lyapunov matrices for system (4.1). The main difficulty that appears in the computation of the matrices as solutions of delay equation (4.3) is the luck of the corresponding initial conditions. To some extent, symmetry condition (4.4) compensates this deficiency, but the computation problem remains complicated.

4.5.1 A Particular Case

In what follows we show that in the case of a polynomial matrix

$$G(\theta)=\sum_{j=1}^{m}\theta^{j-1}B_j, \tag{4.19}$$

where $B_1,\dots,B_m$ are constant $n\times n$ matrices, a Lyapunov matrix $U(\tau)$ may be computed as a solution of an auxiliary delay-free system of linear ordinary differential matrix equations. To this end, we first define the matrices

$$Z(\tau)=U(\tau),\quad V(\tau)=U(\tau-h),\quad \tau\in[0,h],$$

and the set of $2m$ auxiliary matrices

$$X_j(\tau)=\int_{-h}^{0}\theta^{j-1}U(\tau+\theta)\mathrm{d}\theta,\quad Y_j(\tau)=\int_{-h}^{0}\theta^{j-1}U(\tau-\theta-h)\mathrm{d}\theta,\quad j=1,\dots,m.$$

Then Eq. (4.3) can be written as

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}=Z(\tau)A_0+V(\tau)A_1+\sum_{j=1}^{m}X_j(\tau)B_j.$$

Now we compute the first derivative of the matrix $V(\tau)$:

$$\begin{aligned}\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau}&=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(h-\tau)\right]^T\\&=-\left[U(h-\tau)A_0+U(-\tau)A_1+\int_{-h}^{0}\theta^{j-1}U(h-\tau+\theta)\mathrm{d}\theta B_j\right]^T.\end{aligned}$$

Observe that

$$U(h-\tau)=V^T(\tau),\quad U(-\tau)=Z^T(\tau)$$

and

$$\int_{-h}^{0}\theta^{j-1}U(h-\tau+\theta)\mathrm{d}\theta=\left[\int_{-h}^{0}\theta^{j-1}U(\tau-\theta-h)\mathrm{d}\theta\right]^T=Y_j^T(\tau),\quad j=1,2,\dots,m,$$

hence

$$\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau}=-A_0^TV(\tau)-A_1^TZ(\tau)-\sum_{j=1}^{m}B_j^TY_j(\tau).$$

The first derivatives of the matrices $X_1(\tau)$ and $Y_1(\tau)$ are

$$\frac{\mathrm{d}X_1(\tau)}{\mathrm{d}\tau}=U(\tau)-U(\tau-h)=Z(\tau)-V(\tau),$$

$$\frac{\mathrm{d}Y_1(\tau)}{\mathrm{d}\tau}=-V(\tau)+V(\tau+h)=-V(\tau)+Z(\tau).$$

Now, for $j = 2, \dots, m$,

$$\frac{\mathrm{d}X_j(\tau)}{\mathrm{d}\tau} = -(-h)^{j-1}U(\tau - h) - (j-1)\int_{-h}^{0} \theta^{j-2}U(\tau+\theta)\mathrm{d}\theta$$
$$= -(-h)^{j-1}V(\tau) - (j-1)X_{j-1}(\tau)$$

and

$$\frac{\mathrm{d}Y_j(\tau)}{\mathrm{d}\tau} = (-h)^{j-1}U(\tau) + (j-1)\int_{-h}^{0} \theta^{j-2}U(\tau-\theta-h)\mathrm{d}\theta$$
$$= (-h)^{j-1}Z(\tau) + (j-1)Y_{j-1}(\tau).$$

As a result, we arrive at the conclusion that the set of matrices

$$\{Z(\tau), V(\tau), X_1(\tau), \dots, X_m(\tau), Y_1(\tau), \dots, Y_m(\tau)\}$$

satisfies the following delay-free system of $2(m+1)$ ordinary differential matrix equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}Z = ZA_0 + VA_1 + \sum_{j=1}^{m} X_jB_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}V = -A_1^TZ - A_0^TV - \sum_{j=1}^{m} B_j^TY_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_1 = Z - V, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}Y_1 = Z - V, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_j = -(-h)^{j-1}V - (j-1)X_{j-1}, \quad j = 2, \dots, m, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}Y_j = (-h)^{j-1}Z + (j-1)Y_{j-1}, \quad j = 2, \dots, m. \end{cases} \tag{4.20}$$

Lemma 4.2. *The spectrum of system (4.20) is symmetrical with respect to the origin of the complex plane.*

Proof. A complex number s_0 is an eigenvalue of system (4.20) if and only if there exists a nontrivial set of $n \times n$ matrices

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \dots, X_m^{(0)}, Y_1^{(0)}, \dots, Y_m^{(0)}\right\}$$

satisfying the following system of matrix equations:

$$\begin{cases} s_0 Z^{(0)} = Z^{(0)} A_0 + V^{(0)} A_1 + \sum\limits_{j=1}^{m} X_j^{(0)} B_j, \\ s_0 V^{(0)} = -A_1^T Z^{(0)} - A_0^T V^{(0)} - \sum\limits_{j=1}^{m} B_j^T Y_j^{(0)}, \\ s_0 X_1^{(0)} = Z^{(0)} - V^{(0)}, \\ s_0 Y_1^{(0)} = Z^{(0)} - V^{(0)}, \\ s_0 X_j^{(0)} = -(-h)^{j-1} V^{(0)} - (j-1) X_{j-1}^{(0)}, \quad j = 2, \dots, m, \\ s_0 Y_j^{(0)} = (-h)^{j-1} Z^{(0)} + (j-1) Y_{j-1}^{(0)}, \quad j = 2, \dots, m. \end{cases} \tag{4.21}$$

It is easy to check that the matrices

$$\widetilde{Z}^{(0)} = \left(V^{(0)}\right)^T, \quad \widetilde{V}^{(0)} = \left(Z^{(0)}\right)^T, \quad \widetilde{X}_j^{(0)} = \left(Y_j^{(0)}\right)^T,$$
$$\widetilde{Y}_j^{(0)} = \left(X_j^{(0)}\right)^T, \quad j = 1, \dots, m,$$

satisfy the system

$$\begin{cases} -s_0 \widetilde{Z}^{(0)} = \widetilde{Z}^{(0)} A_0 + \widetilde{V}^{(0)} A_1 + \sum\limits_{j=1}^{m} \widetilde{X}_j^{(0)} B_j, \\ -s_0 \widetilde{V}^{(0)} = -A_1^T \widetilde{Z}^{(0)} - A_0^T \widetilde{V}^{(0)} - \sum\limits_{j=1}^{m} B_j^T \widetilde{Y}_j^{(0)}, \\ -s_0 \widetilde{X}_1^{(0)} = \widetilde{Z}^{(0)} - \widetilde{V}^{(0)}, \\ -s_0 \widetilde{Y}_1^{(0)} = \widetilde{Z}^{(0)} - \widetilde{V}^{(0)}, \\ -s_0 \widetilde{X}_j^{(0)} = -(-h)^{j-1} \widetilde{V}^{(0)} - (j-1) \widetilde{X}_{j-1}^{(0)}, \quad j = 2, \dots, m, \\ -s_0 \widetilde{Y}_j^{(0)} = (-h)^{j-1} \widetilde{Z}^{(0)} + (j-1) \widetilde{Y}_{j-1}^{(0)}, \quad j = 2, \dots, m. \end{cases}$$

This means that $-s_0$ belongs to the spectrum of system (4.20). □

The solution of system (4.20) defined by the matrix $U(\tau)$ satisfies also the following set of boundary value conditions:

$$Z(0) = V^T(h),$$

$$X_j(0) = \int_{-h}^{0} \theta^{j-1} U(\theta) \mathrm{d}\theta = \left[\int_{-h}^{0} \theta^{j-1} U(h-\theta-h) \mathrm{d}\theta \right]^T$$
$$= Y_j^T(h), \quad j = 1, \dots, m,$$

$$Y_j(0) = \int_{-h}^{0} \theta^{j-1} U(-\theta-h) \mathrm{d}\theta = \left[\int_{-h}^{0} \theta^{j-1} U(h+\theta) \mathrm{d}\theta \right]^T$$
$$= X_j^T(h), \quad j = 1, \dots, m,$$

as well as the algebraic condition

$$Z(0)A_0 + V(0)A_1 + \sum_{j=1}^{m} X_j(0)B_j + A_0^T V(h) + A_1^T Z(h) + \sum_{j=1}^{m} B_j^T Y_j(h) = -W.$$

We finally arrive at the following statement.

Theorem 4.4. *Given a time-delay system (4.1), where matrix $G(\theta)$ is of the form (4.19), let $U(\tau)$ be a Lyapunov matrix of the system associated with the matrix W. Then the set of matrices*

$$\{Z(\tau), V(\tau), X_1(\tau), \dots, X_m(\tau), Y_1(\tau), \dots, Y_m(\tau)\}$$

is a solution of system (4.20) that satisfies the boundary value conditions

$$\begin{cases} Z(0) = V^T(h), \\ X_j(0) = Y_j^T(h), \text{ and } Y_j(0) = X_j^T(h), \quad j = 1, \dots, m, \\ Z(0)A_0 + V(0)A_1 + \sum_{j=1}^{m} X_j(0)B_j + A_0^T V(h) + A_1^T Z(h) \\ \quad + \sum_{j=1}^{m} B_j^T Y_j(h) = -W. \end{cases} \tag{4.22}$$

There exist some relations between the auxiliary matrices that are described in the following lemma.

Lemma 4.3. *The auxiliary matrices $X_j(\tau)$ and $Y_j(\tau)$, $j = 1, \dots, m$, satisfy the relations*

$$X_j(\tau) = (-1)^{j-1} \sum_{k=0}^{j-1} \frac{(j-1)!}{k!(j-k-1)!} h^k Y_{j-k}(\tau), \quad j = 1, \dots, m,$$

and

$$Y_j(\tau) = (-1)^{j-1}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}h^k X_{j-k}(\tau), \quad j = 1,\dots,m.$$

Proof. The first set of relations can be easily obtained as follows:

$$\begin{aligned}
X_j(\tau) &= \int_{-h}^{0}\theta^{j-1}U(\tau+\theta+h-h)\mathrm{d}\theta \\
&= \langle \xi = -\theta - h\rangle = \int_{-h}^{0}(-h-\xi)^{j-1}U(\tau-\xi-h)\mathrm{d}\xi \\
&= (-1)^{j-1}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}h^k\int_{-h}^{0}\xi^{j-k-1}U(\tau-\xi-h)\mathrm{d}\xi \\
&= (-1)^{j-1}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}h^k Y_{j-k}(\tau).
\end{aligned}$$

The second set of relations can be obtained in a similar way. □

Lemma 4.3 provides a reduction of system (4.20). We have the sum

$$\sum_{j=1}^{m}B_j^T Y_j(\tau) = \sum_{j=1}^{m}(-1)^{j-1}B_j^T\left(\sum_{k=0}^{j-1}\frac{(j-1)!}{k!\,(j-k-1)!}h^k X_{j-k}(\tau)\right).$$

If we define the matrix

$$B(\xi) = \sum_{j=1}^{m}(-\xi)^{j-1}B_j^T,$$

then we obtain the sum

$$\sum_{j=1}^{m}B_j^T Y_j(\tau) = \sum_{k=1}^{m}\left[\frac{1}{(k-1)!}B^{(k-1)}(h)\right]X_k(\tau),$$

where

$$B^{(k-1)}(h) = \left.\frac{\mathrm{d}^{k-1}B(\xi)}{\mathrm{d}\xi^{k-1}}\right|_{\xi=h}, \quad k = 1,2,\dots,m.$$

The second equation of system (4.20) takes the form

$$\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau} = -A_1^T Z(\tau) - A_0^T V(\tau) - \sum_{k=1}^{m}\left[\frac{1}{(k-1)!}B^{(k-1)}(h)\right]X_k(\tau).$$

Therefore, system (4.20) is reduced to the following system of $(m+2)$ matrix equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}Z = ZA_0 + VA_1 + \sum_{j=1}^{m} X_j B_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}V = -A_1^T Z - A_0^T V - \sum_{j=1}^{m} \left[\frac{1}{(k-1)!} B^{(k-1)}(h) \right] X_k(\tau), \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_1 = Z - V, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_j = -(-h)^{j-1}V - (j-1)X_{j-1}, \quad j = 2,\dots,m. \end{cases} \tag{4.23}$$

In a similar way, the set of boundary value conditions (4.22) is reduced to the next one:

$$\begin{cases} Z(0) = V^T(h), \\ X_k(0) = (-1)^{k-1} \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-j-1)!} h^j X_{k-j}^T(h), \quad k = 1,\dots,m, \\ Z(0)A_0 + A_0^T Z(0) + V(0)A_1 + A_1^T V^T(0) + \sum_{j=1}^{m} \left[X_j(0)B_j + B_j^T X_j^T(0) \right] = -W. \end{cases}$$

In the following statement we show that the spectrum of system (4.1) and that of system (4.20) are connected.

Theorem 4.5. *Given a time-delay system (4.1), where the matrix $G(\theta)$ is of the form (4.19), let s_0 be an eigenvalue of the time-delay system such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of delay-free system (4.20).*

Proof. The characteristic matrix of system (4.1) is

$$G(s) = sI - A_0 - \mathrm{e}^{-hs}A_1 - \sum_{k=1}^{m} f^{(k-1)}(s)B_k,$$

where

$$f^{(0)}(s) = \frac{1-\mathrm{e}^{-hs}}{s}, \text{ and } f^{(k-1)}(s) = \frac{\mathrm{d}^{k-1} f(s)}{\mathrm{d}s^{k-1}}, \quad ,k = 2,\dots,m.$$

Because s_0 and $-s_0$ are eigenvalues of the system, there exist nonzero vectors γ and μ such that

$$\gamma^T G(s_0) = 0, \quad G^T(-s_0)\mu = 0. \tag{4.24}$$

A complex number s_0 belongs to the spectrum of system (4.20) if and only if there exists a nontrivial set of $n \times n$ matrices

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \dots, X_m^{(0)}, Y_1^{(0)}, \dots, Y_m^{(0)}\right\}$$

that satisfies (4.21). Multiplying the first equality in (4.24) by μ from the left and the second equality by $-\mathrm{e}^{-hs_0}\gamma^T$ from the right we obtain

$$s_0\mu\gamma^T - \mu\gamma^T A_0 - \mathrm{e}^{-hs_0}\mu\gamma^T A_1 - \sum_{k=1}^{m} f^{(k-1)}(s_0)\mu\gamma^T B_k = 0_{n\times n}$$

and

$$s_0\mathrm{e}^{-hs_0}\mu\gamma^T + A_0^T\mathrm{e}^{-hs_0}\mu\gamma^T + A_1^T\mu\gamma^T + \sum_{k=1}^{m}\mathrm{e}^{-hs_0} f^{(k-1)}(-s_0)B_k^T\mu\gamma^T = 0_{n\times n}.$$

If we introduce the nontrivial matrices

$$Z^{(0)} = \mu\gamma^T,\; V^{(0)} = \mathrm{e}^{-hs_0}\mu\gamma^T,$$

$$X_j^{(0)} = f^{(j-1)}(s_0)\mu\gamma^T,\; Y_j^{(0)} = \mathrm{e}^{-hs_0} f^{(j-1)}(-s_0)\mu\gamma^T, \quad j = 1, \dots, m,$$

then the preceding equalities take the form

$$s_0 Z^{(0)} - Z^{(0)}A_0 - V^{(0)}A_1 - \sum_{k=1}^{m} X_k^{(0)}B_k = 0_{n\times n},$$

$$s_0 V^{(0)} + A_0^T V^{(0)} + A_1^T Z^{(0)} + \sum_{k=1}^{m} B_k^T Y_k^{(0)} = 0_{n\times n}.$$

In other words, the matrices satisfy the first two equations of system (4.21). To verify that these matrices satisfy the remaining $2(m+1)$ matrix equations in (4.21), we multiply the identity

$$sf^{(0)}(s) = 1 - \mathrm{e}^{-hs}$$

by the matrix $\mu\gamma^T$ and set $s = s_0$; then we obtain the equality

$$s_0 X_1^{(0)} = Z^{(0)} - V^{(0)}.$$

Now we compute the derivatives

$$\begin{aligned}\frac{\mathrm{d}^{j-1}}{\mathrm{d}s^{j-1}}\left[sf^{(0)}(s)\right] &= sf^{(j-1)}(s) + (j-1)f^{(j-2)}(s)\\ &= -(-h)^{j-1}\mathrm{e}^{-hs}, \quad j = 2, \dots, m-1.\end{aligned}$$

This means that the following identities hold:

$$sf^{(j-1)}(s) = -(-h)^{j-1}\mathrm{e}^{-hs} - (j-1)f^{(j-2)}(s), \quad j = 2,\ldots,m.$$

If we multiply these identities by the matrix $\mu\gamma^T$ and set $s = s_0$, then we obtain the desired set of matrix equalities

$$s_0 X_j^{(0)} = -(-h)^{j-1}V^{(0)} - (j-1)X_{j-1}^{(0)}, \quad j = 2,\ldots,m.$$

In a similar way one can verify the remaining equalities in (4.21).

It is evident that the set of matrices introduced previously,

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \ldots, X_m^{(0)}, Y_1^{(0)}, \ldots, Y_m^{(0)}\right\},$$

is not trivial. Therefore, the complex value s_0 belongs to the spectrum of system (4.20). The same is true for $-s_0$. □

Remark 4.2. The statement remains valid if we replace in Theorem 4.5 system (4.20) by the reduced system (4.23).

4.5.2 A Special Case

Now we consider the case where the matrix $G(\theta)$ is of the form

$$G(\theta) = \sum_{j=1}^{m} \eta_j(\theta)B_j, \tag{4.25}$$

where $B_1,\ldots,B_m$ are given $n\times n$ matrices and the scalar functions $\eta_1(\theta),\ldots,\eta_m(\theta)$ are such that

$$\frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} = \sum_{k=1}^{m} \alpha_{jk}\eta_k(\theta), \quad j = 1,\ldots,m.$$

Remark 4.3. In the previous subsection we had $\eta_j(\theta) = \theta^{j-1}$, $j = 1,\ldots,m$. These functions satisfy the equations

$$\frac{\mathrm{d}\eta_1(\theta)}{\mathrm{d}\theta} = 0, \quad \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} = (j-1)\,\eta_{j-1}(\theta), \quad j = 2,\ldots,m.$$

The time-delay matrix equation for $U(\tau)$ is now of the form

$$\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = U(\tau)A_0 + U(\tau-h)A_1 + \sum_{j=1}^{m}\int_{-h}^{0} \eta_j(\theta)U(\tau+\theta)B_j\mathrm{d}\theta, \quad \tau \geq 0. \tag{4.26}$$

Let us define for $\tau \in [0,h]$ the matrices $Z(\tau) = U(\tau)$, $V(\tau) = U(\tau - h)$, and

$$X_j(\tau) = \int_{-h}^{0} \eta_j(\theta)U(\tau+\theta)\mathrm{d}\theta, \quad Y_j(\tau) = \int_{-h}^{0} \eta_j(\theta)U(\tau-\theta-h)\mathrm{d}\theta, \quad j = 1,\ldots,m.$$

Then Eq. (4.26) has the form

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau} = Z(\tau)A_0 + V(\tau)A_1 + \sum_{j=1}^{m} X_j(\tau)B_j, \quad \tau \in [0,h],$$

and

$$\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau} = -A_1^T Z(\tau) - A_0^T V(\tau) - \sum_{j=1}^{m} B_j^T Y_j(\tau).$$

Now

$$\begin{aligned}\frac{\mathrm{d}X_j(\tau)}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_{-h}^{0} \eta_j(\theta)U(\tau+\theta)\mathrm{d}\theta\right) \\ &= \eta_j(0)U(\tau) - \eta_j(-h)U(\tau-h) - \int_{-h}^{0} \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta}U(\tau+\theta)\mathrm{d}\theta \\ &= \eta_j(0)Z(\tau) - \eta_j(-h)V(\tau) - \sum_{k=1}^{m} \alpha_{jk}X_k(\tau), \quad j = 1,\ldots,m,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}Y_j(\tau)}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_{-h}^{0} \eta_j(\theta)U(\tau-\theta-h)\mathrm{d}\theta\right) \\ &= -\eta_j(0)U(\tau-h) + \eta_j(-h)U(\tau) + \int_{-h}^{0} \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta}U(\tau-\theta-h)\mathrm{d}\theta \\ &= \eta_j(-h)Z(\tau) - \eta_j(0)V(\tau) + \sum_{k=1}^{m} \alpha_{jk}Y_k(\tau), \quad j = 0,1,\ldots,m.\end{aligned}$$

We arrive at the following system of delay-free matrix equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau} Z = ZA_0 + VA_1 + \sum_{j=1}^{m} X_j B_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau} V = -A_0^T V - A_1^T Z - \sum_{j=1}^{m} B_j^T Y_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau} X_j = \eta_j(0) Z - \eta_j(-h) V - \sum_{k=1}^{m} \alpha_{jk} X_k, \quad j = 1, \dots, m, \\ \frac{\mathrm{d}}{\mathrm{d}\tau} Y_j = \eta_j(-h) Z - \eta_j(0) V + \sum_{k=1}^{m} \alpha_{jk} Y_k, \quad j = 1, \dots, m. \end{cases} \tag{4.27}$$

Because the auxiliary matrices $Z(\tau)$, $V(\tau)$, $X_j(\tau)$, $Y_j(\tau)$, $j = 1, \dots, m$, satisfy the boundary value conditions (4.22), the following result holds.

Theorem 4.6. *Given a time-delay system (4.1), where the matrix $G(\theta)$ is of the form (4.25), let $U(\tau)$ be a Lyapunov matrix of the delay system associated with the matrix W. Then the matrices $Z(\tau)$, $V(\tau)$, $X_j(\tau)$, $Y_j(\tau)$, $j = 1, \dots, m$, define a solution of the auxiliary boundary value problem (4.27), (4.22).*

For the special case the statement of Theorem 4.5 remains true.

Theorem 4.7. *Given a time-delay system (4.1), where the matrix $G(\theta)$ is of the form (4.25), let s_0 be an eigenvalue of the time-delay system such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of system (4.27).*

Sometimes it is possible to perform a reduction of delay-free system (4.27). This happens when the functions $\eta_j(\theta)$, $j = 1, \dots, m$, satisfy the conditions

$$\eta_j(-\theta - h) = \sum_{k=1}^{m} \gamma_{jk} \eta_k(\theta), \quad \theta \in [-h, 0], \quad j = 1, \dots, m.$$

In this case

$$\begin{aligned} Y_j(\tau) &= \int_{-h}^{0} \eta_j(\theta) U(\tau - \theta - h) \mathrm{d}\theta = \langle \xi = -\theta - h \rangle \\ &= \int_{-h}^{0} \eta_j(-\xi - h) U(\tau + \xi) \mathrm{d}\xi = \sum_{k=1}^{m} \gamma_{jk} \int_{-h}^{0} \eta_k(\xi) U(\tau + \xi) \mathrm{d}\xi \\ &= \sum_{k=1}^{m} \gamma_{jk} X_k(\tau), \quad j = 1, \dots, m, \end{aligned}$$

and one can exclude the matrices $Y_j(\tau)$ of system (4.27), as well as those of boundary value conditions (4.22).

4.5.3 Numerical Scheme

In this section we propose a numerical scheme to approximate Lyapunov matrices.

Given a symmetric matrix W, we are looking for an approximate initial condition for the Lyapunov matrix associated with W of the form

$$\Phi(\theta) = \sum_{j=0}^{m} \theta^j \Phi_j, \quad \theta \in [-h, 0],$$

where $\Phi_j,\ j = 0, 1, \ldots, m$, are $n \times n$ constant matrices. We address symmetry property (4.4). According to this property,

$$\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} = (-1)^k \left[\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-0} \right]^T, \quad k \geq 0. \tag{4.28}$$

Here $\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0}$ and $\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-0}$ stand for the right-hand side and the left-hand side derivatives of $U(\tau)$ of the order k at $\tau = 0$, respectively. It follows from (4.3) that

$$\left.\frac{\mathrm{d}^{k+1} U(\tau)}{\mathrm{d}\tau^{k+1}}\right|_{\tau=+0} = \left(\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} \right) A_0 + \left(\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-h+0} \right) A_1$$
$$+ \int_{-h}^{0} \frac{\mathrm{d}^k U(\theta)}{\mathrm{d}\theta^k} G(\theta)\mathrm{d}\theta, \quad k \geq 0.$$

If we replace $U(\theta)$ in the preceding equality by $\Phi(\theta)$, then we obtain that

$$\left.\frac{\mathrm{d}^{k+1} \widehat{U}(\tau)}{\mathrm{d}\tau^{k+1}}\right|_{\tau=+0} = \left(\left.\frac{\mathrm{d}^k \widehat{U}(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} \right) A_0 + \sum_{j=k}^{m} j(j-1)\ldots(j-k-1)\Phi_j$$
$$\times \left[(-h)^{j-k} A_1 + \int_{-h}^{0} \theta^{j-k} G(\theta)\mathrm{d}\theta \right], \quad k \geq 0.$$

For $k = 0$ we have

$$\left.\frac{\mathrm{d}\widehat{U}(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0} = \Phi_0 A_0 + \sum_{j=0}^{m} (-h)^j \Phi_j A_1 + \sum_{j=0}^{m} \Phi_j \int_{-h}^{0} \theta^j G(\theta)\mathrm{d}\theta$$
$$= \sum_{j=0}^{m} \Phi_j L_j^{(1)},$$

where

$$L_0^{(1)} = A_0 + A_1 + \int_{-h}^{0} G(\theta)\mathrm{d}\theta, \quad L_j^{(1)} = (-h)^j A_1 + \int_{-h}^{0} \theta^j G(\theta)\mathrm{d}\theta, \quad j = 1,2,\dots,m.$$

For $k = 1$

$$\begin{aligned}\left.\frac{\mathrm{d}^2\widehat{U}(\tau)}{\mathrm{d}\tau^2}\right|_{\tau=+0} &= \left(\left.\frac{\mathrm{d}\widehat{U}(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0}\right) A_0 + \sum_{j=1}^{m} j\Phi_j \left[(-h)^{j-1}A_1 + \int_{-h}^{0} \theta^{j-1} G(\theta)\mathrm{d}\theta\right] \\ &= \sum_{j=0}^{m} \Phi_j L_j^{(2)},\end{aligned}$$

where

$$L_0^{(2)} = L_0^{(1)} A_0, \quad L_j^{(2)} = L_j^{(1)} A_0 + jL_{j-1}^{(1)}, \quad j = 1,2,\dots,m.$$

On the one hand, repeating this process we obtain the following expressions for the right-hand-side derivatives:

$$\left.\frac{\mathrm{d}^k\widehat{U}(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} = \sum_{j=0}^{m} \Phi_j L_j^{(k)}, \quad k = 1,2,\dots,m.$$

Here

$$L_j^{(k)} = \begin{cases} L_j^{(k-1)} A_0, & j = 0,1,\dots,k-2, \\ L_j^{(k-1)} A_0 + j(j-1)\dots(j-k+2)L_{j-k+1}^{(1)}, & j = k-1,k,\dots,m. \end{cases}$$

On the other hand, the left-hand-side derivatives at $\tau = 0$ are of the form

$$\left.\frac{\mathrm{d}^k\widehat{U}(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-0} = k!\Phi_k, \quad k = 1,2,\dots,m.$$

Substituting these expressions into (4.28) we obtain a system of $(m+1)$ matrix equations for $(m+1)$ matrices Φ_j, $j = 0,1,\dots,m$:

$$\begin{cases} (-1)^k k!\Phi_k^T = \sum_{j=0}^{m} \Phi_j L_j^{(k)}, \quad k = 1,2,\dots,m, \\ \sum_{j=0}^{m} \Phi_j L_j^{(1)} + \Phi_1^T = -W. \end{cases} \tag{4.29}$$

The last equation of this system is property (4.6), written in terms of the matrices.

If system (4.29) admits a solution, Φ_j, $j=0,1,\dots,m$, then we arrive at the matrix

$$\Phi(\theta)=\sum_{j=0}^{m}\theta_j^j\Phi_j,\quad \theta\in[-h,0].$$

The desired approximation of the Lyapunov matrix associated with W is now of the form

$$\widehat{U}(\tau)=[\Phi(-\tau)]^T,\quad \tau\in[0,h].$$

4.6 Complete Type Functionals

Here we define a new class of quadratic functionals. But first we prove the statement.

Theorem 4.8. *Define for the given symmetric matrices W_0, W_1, and W_2 the functional*

$$\begin{aligned} w(\varphi) &= \varphi^T(0)W_0\varphi(0)+\varphi^T(-h)W_1\varphi(-h) \\ &\quad +\int_{-h}^{0}\varphi^T(\theta)W_2\varphi(\theta)\mathrm{d}\theta,\quad \varphi\in PC([-h,0],R^n). \end{aligned} \tag{4.30}$$

Let there exist a Lyapunov matrix $U(\tau)$ associated with matrix

$$W=W_0+W_1+hW_2.$$

This Lyapunov matrix defines the functional $v_0(\varphi)$; see (4.7). The time derivative of the functional

$$v(\varphi)=v_0(\varphi)+\int_{-h}^{0}\varphi^T(\theta)\left[W_1+(h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta,\quad \varphi\in PC([-h,0],R^n), \tag{4.31}$$

along the solutions of system (4.1) is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t)=-w(x_t),\quad t\geq 0.$$

Proof. The proof is similar to that of Theorem 3.4. □

Definition 4.2. We say that functional (4.31) is of the complete type if the matrices W_0, W_1, and W_2 are positive definite.

Lemma 4.4. *Let system (4.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, the complete type functional (4.31) admits a lower bound of the form*

$$\beta_1 \|\varphi(0)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n),$$

where β_1 and β_2 are positive constants.

Proof. We define an auxiliary functional of the form

$$\widetilde{v}(\varphi) = v(\varphi) - \beta_1 \|\varphi(0)\|^2 - \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where β_1 and β_2 are assumed to be positive constants. The time derivative of the functional along the solution of system (4.1) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t),$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\beta_1 x^T(t)\left[A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right] \\ &+ \beta_2 x^T(t)x(t) - \beta_2 x^T(t-h)x(t-h), \quad t \geq 0.\end{aligned}$$

The functional $\widetilde{w}(\varphi)$ admits a lower estimation of the form

$$\widetilde{w}(\varphi) \geq [\varphi^T(0), \varphi^T(-h)]R_1(\beta_1,\beta_2)\begin{bmatrix}\varphi(0)\\ \varphi(-h)\end{bmatrix} + \int_{-h}^{0} \varphi^T(\theta)R_2(\theta,\beta_1)\varphi(\theta)\mathrm{d}\theta,$$

where

$$R_1(\beta_1,\beta_2) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \beta_1 \begin{pmatrix} A_0 + A_0^T - hI & A_1 \\ A_1^T & 0_{n\times n} \end{pmatrix} + \beta_2 \begin{pmatrix} I & 0_{n\times n} \\ 0_{n\times n} & -I \end{pmatrix}$$

and

$$R_2(\theta,\beta_1) = W_2 - \beta_1 G^T(\theta)G(\theta).$$

The matrices W_0, W_1, and W_2 are positive definite, so there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that the following inequalities hold

$$R_1(\beta_1,\beta_2) \geq 0, \quad R_2(\theta,\beta_1) \geq 0, \quad \theta \in [-h,0].$$

For these β_1 β_2 we have

$$\widetilde{w}(\varphi) \geq 0, \quad \varphi \in PC([-h,0],R^n).$$

Therefore,

$$\widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0, \quad \varphi \in PC([-h,0],R^n),$$

and we arrive at the conclusion that

$$\beta_1 \|\varphi(0)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \qquad \square$$

Corollary 4.2. *If we assume $\beta_2 = 0$ and set $\beta_1 = \alpha_1$, then there exists $\alpha_1 > 0$ such that the following inequalities hold:*

$$R_1(\alpha_1,0) \geq 0, \quad R_2(\theta,\alpha_1) \geq 0, \quad \theta \in [-h,0].$$

Therefore, the complete type functional $v(\varphi)$ admits a lower bound of the form

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \tag{4.32}$$

Lemma 4.5. *Let system (4.1) satisfy the Lyapunov condition. Given the symmetric matrices W_0, W_1, and W_2, there exist positive constants δ_1 and δ_2 such that functional (4.31) admits an upper bound of the form*

$$v(\varphi) \leq \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n). \tag{4.33}$$

Proof. The Lyapunov condition implies that there exists a Lyapunov matrix $U(\tau)$ associated with matrix $W = W_0 + W_1 + hW_2$. We define the following constants:

$$\nu = \max_{\tau\in[0,h]} \|U(\tau)\|, \quad a = \|A_1\|, \quad g = \max_{\theta\in[-h,0]} \|G(\theta)\|.$$

Now we estimate the summands that constitute functional (4.31). The sum of the first two terms admits the upper bound

$$\begin{aligned}R_1+R_2 &= \varphi^T(0)U(0)\varphi(0)+2\varphi^T(0)\int_{-h}^{0}U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta\\ &\leq \nu(1+ha)\|\varphi(0)\|^2+\nu a\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.\end{aligned}$$

The sum of the next two terms can be estimated as follows:

$$\begin{aligned}R_3+R_4 &= 2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta}U(\xi-\theta)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta\\ &\quad+\int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0}U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1\\ &\leq 2\nu g\|\varphi(0)\|\int_{-h}^{0}(h+\theta)\|\varphi(\theta)\|\mathrm{d}\theta+\nu a^2\left(\int_{-h}^{0}\|\varphi(\theta)\|\mathrm{d}\theta\right)^2\\ &\leq \nu gh\|\varphi(0)\|^2+\nu h\left(\frac{gh}{3}+a^2\right)\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.\end{aligned}$$

The fifth term admits the estimation

$$\begin{aligned}R_5 &= 2\int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_2}U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right]\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1\\ &\leq 2\nu ag\left(\int_{-h}^{0}\|\varphi(\theta_1)\|\mathrm{d}\theta_1\right)\left(\int_{-h}^{0}(h+\theta_2)\|\varphi(\theta_2)\|\mathrm{d}\theta_2\right)\\ &\leq 2\nu ag\left(\sqrt{h\int_{-h}^{0}\|\varphi(\theta_1)\|^2\mathrm{d}\theta_1}\right)\left(\sqrt{\frac{h^3}{3}\int_{-h}^{0}\|\varphi(\theta_2)\|^2\mathrm{d}\theta_2}\right)\\ &\leq \frac{2}{\sqrt{3}}\nu h^2ag\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.\end{aligned}$$

The next term can be estimated as follows:

$$\begin{aligned} R_6 &= \int_{-h}^{0} \varphi^T(\theta_1) \left\{ \int_{-h}^{0} \left[\int_{-h}^{\theta_1} G^T(\xi_1) \left(\int_{-h}^{\theta_2} U(\theta_1-\theta_2-\xi_1+\xi_2) G(\xi_2) \mathrm{d}\xi_2 \right) \mathrm{d}\xi_1 \right] \right. \\ &\quad \left. \times\ \varphi(\theta_2)\mathrm{d}\theta_2 \right\} \mathrm{d}\theta_1 \\ &\leq \nu g^2 \left(\int_{-h}^{0} (h+\theta_1) \|\varphi(\theta_1)\| \mathrm{d}\theta_1 \right) \left(\int_{-h}^{0} (h+\theta_2) \|\varphi(\theta_2)\| \mathrm{d}\theta_2 \right) \\ &\leq \frac{1}{3} \nu h^3 g^2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta. \end{aligned}$$

And, finally,

$$\begin{aligned} R_7 &= \int_{-h}^{0} \varphi^T(\theta) \left[W_1 + (h+\theta) W_2 \right] \varphi(\theta) \mathrm{d}\theta \\ &\leq (\|W_1\| + h\|W_2\|) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta. \end{aligned}$$

If we collect the estimations, then inequality (4.33) holds for

$$\begin{aligned} \delta_1 &= \nu(1+ha+hg), \\ \delta_2 &= \nu a(1+ha) + \frac{1}{3}\nu g h^2 \left(1+2\sqrt{3}a+hg\right) + \|W_1\| + h\|W_2\|. \end{aligned} \qquad \square$$

Corollary 4.3. *If we assume that $\alpha_2 = \delta_1 + h\delta_2$, then functional (4.31) admits an upper bound of the form*

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n). \tag{4.34}$$

4.7 Exponential Estimates

Lemma 4.6. *Given the positive-definite matrices W_0, W_1, and W_2, functional (4.30) admits the following exponential estimate:*

$$\lambda_{\min}(W_0)\,\|\varphi(0)\|^2 + \lambda_{\min}(W_2)\int_{-h}^{0}\|\varphi(\theta)\|^2\,\mathrm{d}\theta \le w(\varphi), \quad \varphi \in PC([-h,0],R^n).$$

Proof. The proof follows directly from (4.30). □

Lemma 4.7. *Let system (4.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, there exists $\sigma > 0$ such that the complete type functional (4.31) satisfies the inequality*

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t} + 2\sigma v(x_t) \le 0, \quad t \ge 0, \tag{4.35}$$

along the solutions of the system.

Proof. On the one hand, by Lemma 4.5, there exist positive constants δ_1 and δ_2 such that

$$v(\varphi) \le \delta_1\,\|\varphi(0)\|^2 + \delta_2\int_{-h}^{0}\|\varphi(\theta)\|^2\,\mathrm{d}\theta.$$

On the other hand, Lemma 4.6 provides the estimate

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t} = -w(x_t) \le -\lambda_{\min}(W_0)\,\|x(t)\|^2 - \lambda_{\min}(W_2)\int_{-h}^{0}\|x(t+\theta)\|^2\,\mathrm{d}\theta.$$

Therefore, any $\sigma > 0$ that satisfies the inequalities

$$2\sigma\delta_1 \le \lambda_{\min}(W_0) \text{ and } 2\sigma\delta_2 \le \lambda_{\min}(W_2)$$

also satisfies (4.35). □

Theorem 4.9. *Let system (4.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, the inequality*

$$\|x(t,\varphi)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\,\|\varphi\|_h\,\mathrm{e}^{-\sigma t}, \quad t \ge 0,$$

holds for any solution of the system. Here α_1 and α_2 are as defined in Corollaries 4.2 and 4.3, respectively, and $\sigma > 0$ is as computed in Lemma 4.7.

Proof. Let $\sigma > 0$ satisfy Lemma 4.7. Then, integrating inequality (4.35), we obtain that

$$v(x_t(\varphi)) \le v(\varphi)\mathrm{e}^{-2\sigma t}, \quad t \ge 0.$$

Now inequalities (4.32) and (4.34) imply that

$$\alpha_1 \|x(t,\varphi)\|^2 \leq v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t} \leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

The desired exponential estimate is a direct consequence of the preceding inequalities. □

Chapter 5
General Theory

This chapter starts the second part of the book, where neutral type time-delay systems are studied. Issues related to the existence, uniqueness, and continuation of solutions of an initial value problem for such systems are discussed. In addition, stability concepts and basic stability results obtained with the use of the Lyapunov–Krasovskii approach, mainly in the form of necessary and sufficient conditions, are presented here.

5.1 System Description

We consider a neutral type time-delay system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] = f(t, x_t). \tag{5.1}$$

Here the functional $f(t,\varphi)$ is defined for $t \in [0,\infty)$ and $\varphi \in PC^1\left([-h,0], R^n\right)$,

$$f : [0,\infty) \times PC^1\left([-h,0], R^n\right) \longrightarrow R^n,$$

and is continuous in both arguments. The matrix D is a given $n \times n$ matrix, delay $h > 0$. The information needed to begin the computation of a particular solution of the system includes an initial time instant $t_0 \geq 0$ and an initial function $\varphi : [-h,0] \rightarrow R^n$, and it is assumed that

$$x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-h,0]. \tag{5.2}$$

As usual, the state of the system at the time instant $t \geq t_0$ is defined as the restriction,

$$x_t : \theta \rightarrow x(t+\theta), \quad \theta \in [-h,0],$$

V.L. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*, Control Engineering, DOI 10.1007/978-0-8176-8367-2_5,

of the solution $x(t)$ on the segment $[t-h,t]$. If the initial condition (t_0,φ) is indicated explicitly, then we use the notations $x(t,t_0,\varphi)$ and $x_t(t_0,\varphi)$. In the case of time-invariant systems we usually assume that $t_0=0$ and omit the argument t_0 in these notations.

We will use initial functions from the space $PC^1([-h,0],R^n) \subset PC([-h,0],R^n)$. Here it is assumed that a function $\varphi \in PC([-h,0],R^n)$ belongs to $PC^1([-h,0],R^n)$ if on each continuity interval $(\alpha,\beta) \in [-h,0]$ the function is continuously differentiable and the first derivative of the function, $\varphi'(\theta)$, has a finite right-hand-side limit at $\theta=\alpha$, $\varphi'(\alpha+0)=\lim_{\varepsilon\to 0}\varphi'(\alpha+|\varepsilon|)$, and a finite left-hand-side limit at $\theta=\beta$, $\varphi'(\beta-0)=\lim_{\varepsilon\to 0}\varphi'(\beta-|\varepsilon|)$. On the one hand, such a choice creates certain technical difficulties. But on the other hand, it provides several advantages in the formulations and proofs of some statements presented in the chapter. In particular, it follows from Theorem 5.1 that if $\varphi \in PC^1([-h,0],R^n)$, then $x_t(t_0,\varphi) \in PC^1([-h,0],R^n)$ for $t>t_0$.

Henceforth we assume that the following assumptions hold.

Assumption 5.1. *The difference $x(t,t_0,\varphi)-Dx(t-h,t_0,\varphi)$ is continuous and differentiable for $t \geq t_0$, except possibly for a countable number of points. This does not imply that $x(t,t_0,\varphi)$ is differentiable, or even continuous, for $t \geq t_0$.*

Assumption 5.2. *In Eq. (5.1) the right-hand-side derivative of the difference $x(t,t_0,\varphi)-Dx(t-h,t_0,\varphi)$ is assumed at the point $t=t_0$. By default, such agreement remains valid in situations where only a one-sided variation of the independent variable is allowed.*

Let $x(t)$ be a solution of the initial value problem (5.1)–(5.2); then

$$x(t)=Dx(t-h)+[\varphi(0)-D\varphi(-h)]+\int_{t_0}^{t} f(s,x_s)\mathrm{d}s, \quad t \geq t_0. \tag{5.3}$$

System (5.3) is the integral form of the initial value problem. In some sense it is more convenient to consider the integral system than the original one. For example, the choice of $PC([-h,0],R^n)$ as the space of initial functions for system (5.3) seems natural. The integral form substantially simplifies the study of discontinuity points of the solutions of system (5.1). If $\theta_1 \in [-h,0]$ is a discontinuity point of φ, then, according to Assumption 5.1, the function

$$z(t)=Dx(t-h)+[\varphi(0)-D\varphi(-h)]$$

has a jump discontinuity at $t_1=t_0+\theta_1+h$ and the size of the jump at the point is such that

$$\Delta x(t_1)=D\Delta\varphi(\theta_1),$$

where $\Delta x(t_1) = x(t_1+0) - x(t_1-0)$. If $x(t)$ is defined for $t \in [t_0-h,\infty)$, then, as follows from Eq. (5.3), the solution suffers a jump discontinuity at the points $t_k = t_0 + \theta_1 + kh$, $k \geq 0$, and the jumps are subjected to the equation

$$\Delta x(t_{k+1}) = D\Delta x(t_k), \quad k \geq 0.$$

One of the special features of neutral type time-delay systems is the following. The discontinuity of a solution results in the discontinuity of the derivative on the left-hand side of system (5.1). Indeed, consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] = F(x(t), x(t-h)).$$

If $\theta_1 \in [-h,0]$ is a discontinuity point of φ, then

$$\lim_{t\to t_1-0} \frac{\mathrm{d}}{\mathrm{d}t}\left[x(t,\varphi) - Dx(t-h,\varphi)\right] = F\left(x(t_1-0,\varphi), \varphi(\theta_1-0)\right)$$

and

$$\lim_{t\to t_1+0} \frac{\mathrm{d}}{\mathrm{d}t}\left[x(t,\varphi) - Dx(t-h,\varphi)\right] = F\left(x(t_1-0,\varphi) + \Delta x(t_1,\varphi), \varphi(\theta_1-0) + \Delta\varphi(\theta_1)\right).$$

This means that the left-hand-side and right-hand-side derivatives at $t = t_1$ may not coincide. The following assumption makes it possible to overcome this technical difficulty.

Assumption 5.3. *It is assumed that $x(t,t_0,\varphi)$, $t \in [t_0-h, t_0+\tau]$, where $\tau > 0$, is a solution of system (5.1) if it satisfies the system almost everywhere on $[t_0, t_0+\tau]$.*

5.2 Existence Issue

We start with the following existence result.

Theorem 5.1. *Let the functional*

$$f : [0,\infty) \times PC^1\left([-h,0], R^n\right) \longrightarrow R^n$$

satisfy the following conditions:

(i) For any $H > 0$ there exists $M(H) > 0$ such that

$$\|f(t,\varphi)\| \leq M(H), \quad (t,\varphi) \in [0,\infty) \times PC^1\left([-h,0], R^n\right),\ \|\varphi\|_h \leq H.$$

(ii) *The functional $f(t,\varphi)$ is continuous with respect to both arguments.*
(iii) *The functional $f(t,\varphi)$ is Lipschitz with respect to the second argument, i.e., for any $H>0$ there exists a Lipschitz constant $L(H)$ such that the inequality*

$$\left\| f(t,\varphi^{(1)}) - f(t,\varphi^{(2)}) \right\| \le L(H) \left\| \varphi^{(1)} - \varphi^{(2)} \right\|_h$$

holds for $t \ge 0$, $\varphi^{(k)} \in PC^1([-h,0],R^n)$, and $\left\|\varphi^{(k)}\right\|_h \le H$, $k=1,2$.
Then, for given $t_0 \ge 0$ and an initial function $\varphi \in PC^1([-h,0],R^n)$ there exists $\tau > 0$ such that the initial value problem (5.1)–(5.2) admits a unique solution defined on the segment $[t_0-h, t_0+\tau]$.

Proof. Given $t_0 \ge 0$ and $\varphi \in PC^1([-h,0],R^n)$, we introduce the function

$$z(t) = D\varphi(t-t_0-h) + \varphi(0) - D\varphi(-h), \quad t \in [t_0, t_0+h].$$

Let us select $H>0$ such that the following inequality holds:

$$H > H_0 = \max\left\{ \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|, \sup_{t\in[t_0,t_0+h]} \|z(t)\| \right\}.$$

Now we can define the corresponding values $M = M(H)$ and $L = L(H)$; see conditions *(i)* and *(iii)* of the theorem.

Let $\tau \in (0,h)$ be such that

$$\tau L < 1 \text{ and } \tau M < H - H_0.$$

Denote by Θ the set of discontinuity points of the initial function φ, and define a piecewise continuous function $u : [t_0-h, t_0+\tau] \to R^n$ as follows:

$$u(t_0+\theta) = \varphi(\theta), \quad \theta \in [-h,0],$$

and any discontinuity point $t^* \in (t_0, t_0+\tau]$ of the function is such that $t^*-t_0-h \in \Theta$. Finally, assume that the following inequality holds:

$$\|u(t) - z(t)\| \le (t-t_0)M, \quad t \in [t_0, t_0+\tau].$$

The preceding inequality implies that

$$\|u(t)\| \le H_0 + \tau M < H, \quad t \in [t_0, t_0+\tau].$$

It follows from the definition of the function that

$$\|u(t)\| \le H_0 < H, \quad t \in [t_0-h, t_0].$$

We denote by U the set of all such functions. On the set U we define an operator $\mathcal{A}$ that acts on the functions of the set

$$\mathcal{A}(u)(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h,t_0], \\ z(t) + \int\limits_{t_0}^{t} f(s,u_s)\mathrm{d}s, & t \in [t_0,t_0+\tau]. \end{cases}$$

Here $u_s : \theta \to u(s+\theta), \quad \theta \in [-h,0]$ and $\|u_s\|_h \leq H$ for $s \in [t_0,t_0+\tau]$. It is easy to verify that the theorem conditions *(i)* and *(ii)* guarantee that the transformed function, $\mathcal{A}(u)$, belongs to the same set U,

$$u \in U \Rightarrow \mathcal{A}(u) \in U.$$

Any solution $\widetilde{x}(t)$ of the initial value problem (5.1)–(5.2) defines a fixed point of the operator,

$$\widetilde{x}(t) = \mathcal{A}(\widetilde{x})(t), \quad t \in [t_0-h,t_0+\tau].$$

Observe that

$$\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) = \begin{cases} 0, & t \in [t_0-h,t_0], \\ \int\limits_{t_0}^{t} [f(s,u_s^{(1)}) - f(s,u_s^{(2)})]\mathrm{d}s, & t \in [t_0,t_0+\tau]. \end{cases}$$

Hence, for $t \in [t_0-h,t_0]$

$$\left\|\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t)\right\| = 0$$

and for $t \in [t_0,t_0+\tau]$

$$\left\|\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t)\right\| \leq \int\limits_{t_0}^{t_0+\tau} \left\|f(s,u_s^{(1)}) - f(s,u_s^{(2)})\right\| \mathrm{d}s.$$

Because $\|u_s^{(1)}\|_h \leq H$ and $\|u_s^{(2)}\|_h \leq H$, the Lipschitz condition *(iii)* implies that the inequality

$$\begin{aligned} \left\|\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t)\right\| &\leq L \int\limits_{t_0}^{t_0+\tau} \left\|u_s^{(1)} - u_s^{(2)}\right\|_h \mathrm{d}s \\ &\leq \tau L \sup_{s\in[t_0-h,t_0+\tau]} \left\|u^{(1)}(s) - u^{(2)}(s)\right\| \end{aligned}$$

holds for $t \in [t_0, t_0+\tau]$. Since the preceding inequality holds for all $t \in [t_0-h, t_0+\tau]$, we conclude that

$$\sup_{s\in[t_0-h,t_0+\tau]} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| \leq \tau L \sup_{s\in[t_0-h,t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\|.$$

Now, because $L\tau < 1$, the operator $\mathcal{A}$ satisfies the conditions of the contraction mapping theorem, and there exists a unique fixed point of the operator $u^{(*)} \in U$. This means that

$$u^{(*)}(t) = \mathcal{A}(u^{(*)})(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h, t_0], \\ z(t) + \int\limits_{t_0}^{t} f(s, u_s^{(*)})\mathrm{d}s, & t \in [t_0, t_0+\tau], \end{cases}$$

i.e.,

$$u^{(*)}(t) - Du^{(*)}(t-h) = \varphi(0) - D\varphi(-h) + \int\limits_{t_0}^{t} f(s, u_s^{(*)})\mathrm{d}s, \quad t \in [t_0, t_0+\tau].$$

The functional $f(t,\varphi)$ is continuous, and $u^{(*)}(t)$ is piecewise continuous; therefore, the right-hand side of the last equality is differentiable on $[t_0, t_0+\tau]$, except at most a finite number of points, and we arrive at the conclusion that the following equality holds almost everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u^{(*)}(t) - Du^{(*)}(t-h)\right] = f(t, u_t^{(*)}), \quad t \in [t_0, t_0+\tau].$$

Because function $u^{(*)}(t)$ satisfies Eq. (5.2), it is the unique solution of the initial value problem (5.1)–(5.2). ☐

Remark 5.1. We can take $t_1 = t_0 + \tau$ as a new initial time instant and define the new initial function

$$\varphi^{(1)}(\theta) = u^{(*)}(t_1+\theta), \quad \theta \in [-h, 0].$$

Then the construction process can be repeated, and we extend the solution to the next segment $[t_1, t_1+\widetilde{\tau}]$. This extension process can be continued as far as the solution remains bounded.

For each solution there exists a maximal interval $[t_0, t_0+T)$ on which the solution is defined. Here we present conditions under which any solution of system (5.1) is defined on the interval $[t_0, \infty)$.

Theorem 5.2. *Let system (5.1) satisfy the conditions of Theorem 5.1. Assume additionally that $f(t,\varphi)$ satisfies the inequality*

$$\|f(t,\varphi)\| \leq \eta(\|\varphi\|_h), \quad t \geq 0, \ \varphi \in PC^1\left([-h,0], R^n\right),$$

where the function $\eta(r)$, $r \in [0,\infty)$, *is continuous, nondecreasing, and such that for any* $r_0 \geq 0$ *the following condition holds:*

$$\lim_{R\to\infty} \int_{r_0}^{R} \frac{\mathrm{d}r}{\eta(r)} = \infty.$$

Then any solution $x(t,t_0,\varphi)$ *of the system is defined on* $[t_0,\infty)$.

Proof. Given $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, there exists a maximal interval $[t_0,t_0+T)$ on which the corresponding solution $x(t,t_0,\varphi)$ is defined. For the sake of simplicity we denote $x(t,t_0,\varphi)$ by $x(t)$.

Denote by $[t_0,t_0+T)$ the maximal interval on which the solution is defined. Assume by contradiction that $T < \infty$, and define the smallest entire N such that $T \leq Nh$. There exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} t_k = t_0 + T$$

and

$$\lim_{k\to\infty} \|x(t_k)\| \to \infty.$$

Otherwise, by Remark 5.1, the solution can be defined on a wider segment $[t_0, t_0 + T + \tau]$, $\tau > 0$.

The solution satisfies the equality

$$x(t) = Dx(t-h) + [\varphi(0) - D\varphi(-h)] + \int_{t_0}^{t} f(s,x_s)\mathrm{d}s, \quad t \in [t_0, t_0+T).$$

For a given $t \in [t_0,t_0+T)$ we define an integer k such that $t \in [t_0+(k-1)h, t_0+kh)$. Now, iterating the preceding equality $k-1$ times, we obtain that

$$x(t) = D^k x(t-kh) + \sum_{j=0}^{k-1} D^j[\varphi(0) - D\varphi(-h)] + \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} f(s,x_s)\mathrm{d}s.$$

There exist $d \geq 1$ and $\rho > 0$ such that $\left\|D^k\right\| \leq d\rho^k$ for $k \geq 0$. Thus

$$\|x(t)\| \leq \kappa \|\varphi\|_h + \varkappa \int_{t_0}^{t} \|f(s,x_s)\| \mathrm{d}s, \quad t \in [t_0,t_0+T),$$

where

$$\varkappa = d\sum_{j=0}^{N-1} \rho^j, \quad \kappa = \max\left\{d, d\rho^N\right\} + (1+\rho)\varkappa.$$

For $\theta \in [-h, 0]$ the following inequality holds:

$$\|x(t+\theta)\| \le \kappa \|\varphi\|_h + \varkappa \int_{t_0}^{\max\{t+\theta, t_0\}} \|f(s, x_s)\| \, \mathrm{d}s$$
$$\le \kappa \|\varphi\|_h + \varkappa \int_{t_0}^{t} \|f(s, x_s)\| \, \mathrm{d}s;$$

hence we arrive at the inequality

$$\|x_t\|_h \le \kappa \|\varphi\|_h + \varkappa \int_{t_0}^{t} \|f(s, x_s)\| \, \mathrm{d}s, \quad t \in [t_0, t_0 + T).$$

It follows from the theorem conditions that

$$\|x_t\|_h \le \kappa \|\varphi\|_h + \varkappa \int_{t_0}^{t} \eta(\|x_s\|_h) \mathrm{d}s, \quad t \in [t_0, t_0 + T).$$

Denote the right-hand side of the last inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \varkappa \eta(\|x_t\|_h) \le \varkappa \eta(v(t)), \quad t \in [t_0, t_0 + T).$$

This implies that

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} \le \varkappa (t_k - t_0), \quad k = 1, 2, 3, \dots.$$

On the one hand, since

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \int_{r_0}^{r_k} \frac{\mathrm{d}\xi}{\eta(\xi)},$$

where $r_0 = v(t_0) = \kappa \|\varphi\|_h \ge 0$, and

$$r_k = v(t_k) \ge \left\|x_{t_k}\right\|_h \ge \|x(t_k)\| \to \infty, \text{ as } k \to \infty,$$

we conclude that

$$\lim_{k \to \infty} \int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \infty.$$

On the other hand,

$$\lim_{k\to\infty} \varkappa(t_k - t_0) = \varkappa T;$$

therefore $T = \infty$. This contradicts our assumption that $T < \infty$. The contradiction concludes the proof of the theorem. □

5.3 Continuity of Solutions

In this section we analyze the continuity properties of the solutions of system (5.1) with respect to initial conditions as well as the system right-hand-side perturbations. These continuity properties are a direct consequence of the following theorem.

Theorem 5.3. *Assume that the right-hand side of system (5.1), $f(t,\varphi)$, satisfies the conditions of Theorem 5.1. Let $x(t,t_0,\varphi)$ be a solution of system (5.1) with the initial condition*

$$x(t_0+\theta) = \varphi(\theta), \quad \theta \in [-h,0].$$

Given a perturbed system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - Dy(t-h)\right] = f(t,y_t) + g(t,y_t), \quad t \geq 0,$$

where the functional $g(t,\varphi)$ is continuous on the set $[0,\infty)\times PC^1([-h,0],R^n)$, the functional $g(t,\varphi)$ satisfies the Lipschitz condition with respect to the second argument and

$$\|g(t,\varphi)\| \leq m, \quad t \geq 0,\ \varphi \in PC^1([-h,0],R^n).$$

Let $y(t,t_0,\psi)$ be a solution of the perturbed system with the initial condition

$$y(t_0+\theta) = \psi(\theta),\ \theta \in [-h,0].$$

If both solutions are defined for $t \in [t_0,t_0+T]$, where $0 < T < \infty$, then there exist positive constants α,β,γ such that the following inequality holds:

$$\|x(t,t_0,\varphi) - y(t,t_0,\psi)\| \leq \left(\alpha\|\psi-\varphi\|_h + \beta m\right)\mathrm{e}^{\gamma(t-t_0)}, \quad t \in [t_0,t_0+T].$$

Proof. For the matrix D there exist $d \geq 1$ and $\rho > 0$ such that $\left\|D^k\right\| \leq d\rho^k$ for $k \geq 0$.

For the sake of simplicity we will use the following shorthand notations for the solutions $x(t,t_0,\varphi) = x(t)$ and $y(t,t_0,\psi) = y(t)$. Observe that for $t \geq t_0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - Dy(t-h)\right] = f(t,x_t) - f(t,y_t) - g(t,y_t).$$

Integrating the preceding equality we obtain that

$$\begin{aligned} x(t)-y(t) &= D\left[x(t-h)-y(t-h)\right] \\ &\quad + \left[\varphi(0)-D\varphi(-h)\right] - \left[\psi(0)-D\psi(-h)\right] \\ &\quad + \int_{t_0}^{t} \left[f(s,x_s)-f(s,y_s)-g(s,y_s)\right] \mathrm{d}s, \quad t \geq t_0. \end{aligned}$$

Let us first define the smallest integer N such that $T \leq hN$. Then for a given $t \in [t_0, t_0+T]$ we define an integer k such that $t \in [t_0+(k-1)h, t_0+kh)$. Now, after $k-1$ iterations we arrive at the equality

$$\begin{aligned} x(t)-y(t) &= D^k\left[x(t-kh)-y(t-kh)\right] + \sum_{j=0}^{k-1} D^j \left[\varphi(0)-\psi(0)\right] \\ &\quad - \sum_{j=0}^{k-1} D^{j+1} \left[\varphi(-h)-\psi(-h)\right] \\ &\quad + \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} \left[f(s,x_s)-f(s,y_s)-g(s,y_s)\right] \mathrm{d}s. \end{aligned} \tag{5.4}$$

Since $t-kh \in [t_0-h, t_0]$, we conclude that

$$\left\| D^k\left[x(t-kh)-y(t-kh)\right] \right\| \leq d\rho^k \left\|\varphi-\psi\right\|_h \leq \max\left\{d, d\rho^N\right\} \left\|\varphi-\psi\right\|_h.$$

It is obvious that the following two inequalities hold:

$$\left\| \sum_{j=0}^{k-1} D^j \left[\varphi(0)-\psi(0)\right] \right\| \leq \varkappa \left\|\varphi-\psi\right\|_h,$$

where

$$\varkappa = d \sum_{j=0}^{N-1} \rho^j$$

and

$$\left\| \sum_{j=0}^{k-1} D^{j+1} \left[\varphi(-h)-\psi(-h)\right] \right\| \leq \varkappa\rho \left\|\varphi-\psi\right\|_h.$$

Finally, we find that

$$\left\| \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} [f(s,x_s) - f(s,y_s)]\,\mathrm{d}s \right\| \le \varkappa \int_{t_0}^{t} \|f(s,x_s) - f(s,y_s)\|\,\mathrm{d}s$$
$$\le \varkappa L_1 \int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s$$

and

$$\left\| \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} g(s,y_s)\mathrm{d}s \right\| \le \varkappa \int_{t_0}^{t} \|g(s,y_s)\|\,\mathrm{d}s \le \varkappa m(t-t_0).$$

Here $L_1 = L(H_1)$ and

$$H_1 = \max\left\{ \sup_{t\in[t_0-h,t_0+T]} \|x(t)\|,\ \sup_{t\in[t_0-h,t_0+T]} \|y(t)\| \right\}.$$

Now equality (5.4) implies that for $t \in [t_0, t_0+T]$ the inequality

$$\|x(t) - y(t)\| \le \kappa \|\varphi - \psi\|_h + \varkappa m(t-t_0) + \varkappa L_1 \int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s$$

holds, where

$$\kappa = \max\left\{d, d\rho^N\right\} + \varkappa(1+\rho).$$

Applying arguments similar to that used in the proof of Theorem 5.2 we obtain that

$$\|x_t - y_t\|_h \le \kappa \|\varphi - \psi\|_h + \varkappa m(t-t_0) + \varkappa L_1 \int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s, \quad t \in [t_0, t_0+T].$$

Denote the right-hand side of the last inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \varkappa m + \varkappa L_1 \|x_t - y_t\|_h, \quad t \in [t_0, t_0+T].$$

Direct integration of this inequality leads to the desired result

$$\begin{aligned}\|x(t,t_0,\varphi) - y(t,t_0,\psi)\| &\le \|x_t(t_0,\varphi) - y_t(t_0,\psi)\|_h \\ &\le \kappa \|\psi - \varphi\|_h \mathrm{e}^{\varkappa L_1(t-t_0)} + \frac{m}{L}\mathrm{e}^{\varkappa L_1(t-t_0)} \\ &\le (\alpha \|\psi - \varphi\|_h + \beta m)\,\mathrm{e}^{\gamma(t-t_0)}, \quad t \in [t_0, t_0+T],\end{aligned}$$

where $\alpha = \kappa$, $\beta = L_1^{-1}$, and $\gamma = \varkappa L_1$. □

Corollary 5.1. *Let $g(t,\varphi) \equiv 0$, then $m = 0$, and both $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ are solutions of system (5.1). Assume that these solutions are defined for $t \in [t_0, t_0+T]$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\psi - \varphi\|_h < \delta$, then the following inequality holds:*

$$\|x(t,t_0,\varphi) - x(t,t_0,\psi)\| < \varepsilon, \quad t \in [t_0, t_0+T].$$

In other words, $x(t,t_0,\varphi)$ depends continuously on the initial function φ.

Proof. The statement follows directly from Theorem 5.3 if we set $\delta = \varepsilon\alpha^{-1}\mathrm{e}^{-\gamma T}$.□

Corollary 5.2. *Let $\psi(\theta) = \varphi(\theta)$, $\theta \in [-h,0]$; then the solutions $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ have the same initial function. Assume that these solutions are defined for $t \in [t_0,t_0+T]$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m < \delta$, then the following inequality holds:*

$$\|x(t,t_0,\varphi) - y(t,t_0,\varphi)\| < \varepsilon, \quad t \in [t_0, t_0+T].$$

This means that the solutions depend continuously on the right-hand side of system (5.1).

Proof. The statement follows directly from Theorem 5.3 if we set $\delta = \varepsilon\beta^{-1}\mathrm{e}^{-\gamma T}$.□

5.4 Stability Concepts

In the rest of the chapter we assume that system (5.1) satisfies the conditions of Theorem 5.1 and additionally that it admits the trivial solution, i.e., the following identity holds:

$$f(t,0_h) \equiv 0, \text{ for } t \geq 0.$$

Definition 5.1. The trivial solution of system (5.1) is said to be stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon,t_0) > 0$ such that for every initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \delta(\varepsilon,t_0)$, the following inequality holds:

$$\|x(t,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

If $\delta(\varepsilon,t_0)$ can be chosen independently of t_0, then the trivial solution is said to be uniformly stable.

Definition 5.2. The trivial solution of system (5.1) is said to be asymptotically stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\Delta(\varepsilon,t_0) > 0$ such that for every initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon,t_0)$, the following conditions hold.

1. $\|x(t,t_0,\varphi)\| < \varepsilon$, for $t \geq t_0$.
2. $x(t,t_0,\varphi) \to 0$, as $t - t_0 \to \infty$.
   If $\Delta(\varepsilon,t_0)$ can be chosen independently of t_0 and there exists $H_1 > 0$ such that $x(t,t_0,\varphi) \to 0$, because $t - t_0 \longrightarrow \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$, then the trivial solution is said to be uniformly asymptotically stable.

Definition 5.3. The trivial solution of system (5.1) is said to be exponentially stable if there exist $\Delta_0 > 0$, $\sigma > 0$, and $\gamma \geq 1$ such that for every $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \Delta_0$, the following inequality holds:

$$\|x(t,t_0,\varphi)\| \leq \gamma \mathrm{e}^{-\sigma(t-t_0)} \|\varphi\|_h, \quad t \geq t_0.$$

As mentioned in Sect. 5.1, if an initial function φ admits a jump point θ_1, then the corresponding solution, $x(t,t_0,\varphi)$, has jump discontinuity at the points $t_k = t_0 + \theta_1 + kh$, $k \geq 1$, and the jumps at these points satisfy the jump equation

$$\Delta x(t_{k+1}) = D\Delta x(t_k), \quad k \geq 1.$$

As a consequence, we observe that system (5.1) cannot be stable if the matrix D admits an eigenvalue with magnitude greater than one. Otherwise, for any $\delta > 0$ there exists an initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \delta$, such that the corresponding solution $x(t,t_0,\varphi)$ has a sequence of jumps, and the size of the jumps tends to infinity. This observation motivates the following assumption.

Assumption 5.4. *In the rest of the chapter we assume that matrix D is Schur stable, i.e., the spectrum of the matrix lies in the open unit disc of the complex plane.*

5.5 Lyapunov–Krasovskii Approach

We will use the following concept of positive-definite functionals for system (5.1).

Definition 5.4. The functional $v(t,\varphi)$ is said to be positive definite if there exists $H > 0$ such that the following conditions are satisfied:

1. The functional $v(t,\varphi)$ is defined for $t \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.
2. $v(t,0_h) = 0$, $t \geq 0$.
3. There exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0) - D\varphi(-h)) \leq v(t,\varphi),$$
$$t \geq 0,\ \varphi \in PC^1([-h,0],R^n),\ \text{with } \|\varphi\|_h \leq H.$$

4. For any given $t_0 \geq 0$ the functional $v(t_0, \varphi)$ is continuous in φ at the point 0_h, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|\varphi\|_h < \delta$ implies

$$|v(t_0, \varphi) - v(t_0, 0_h)| = v(t_0, \varphi) < \varepsilon.$$

We are now ready to present some basic results of the Lyapunov–Krasovskii approach.

Theorem 5.4. *The trivial solution of system (5.1) is stable if and only if there exists a positive-definite functional $v(t, \varphi)$ such that along the solutions of the system $v(t, x_t)$, as a function of t, does not increase.*

Proof. Sufficiency: Since the matrix D is Schur stable, there exist $d \geq 1$and $\rho \in (0,1)$ such that the inequality $\left\|D^k\right\| \leq d\rho^k$ holds for $k \geq 0$. The positive definiteness of the functional $v(t, \varphi)$ implies that there exists a positive-definite function $v_1(x)$ satisfying Definition 5.4. Let $H > 0$ be that of Definition 5.4.

For a given $\varepsilon \in (0, H)$ we first set

$$\varepsilon_1 = \frac{1-\rho}{d}\varepsilon > 0$$

and then introduce the positive value

$$\lambda(\varepsilon_1) = \min_{\|x\|=\varepsilon_1} v_1(x). \tag{5.5}$$

Since for a given $t_0 \geq 0$ functional $v(t_0, \varphi)$ is continuous in φ at the point 0_h, there exists $\delta_1(\varepsilon, t_0) > 0$ such that $v(t_0, \varphi) < \lambda(\varepsilon_1)$ for any $\varphi \in PC^1([-h, 0], R^n)$, with $\|\varphi\|_h \leq \delta_1(\varepsilon, t_0)$.

It is clear that $\delta_1(\varepsilon, t_0) \leq \varepsilon_1$; otherwise we can present an initial function $\varphi \in PC^1([-h, 0], R^n)$ such that $\|\varphi\|_h < \delta_1(\varepsilon, t_0)$ and $\|\varphi(0) - D\varphi(-h)\| = \varepsilon_1$. On the one hand, for this initial function we have $v_1(\varphi(0) - D\varphi(-h)) \geq \lambda(\varepsilon_1)$. On the other hand, $v_1(\varphi(0) - D\varphi(-h)) \leq v(t_0, \varphi) < \lambda(\varepsilon_1)$. The contradiction proves the desired inequality.

Now we define the positive value

$$\delta(\varepsilon, t_0) = \frac{\delta_1(\varepsilon, t_0)}{1 + d\rho}.$$

Let $\varphi \in PC^1([-h, 0], R^n)$ with $\|\varphi\|_h < \delta(\varepsilon, t_0)$. Then the theorem condition implies that

$$\begin{aligned} v_1(x(t, t_0, \varphi) - Dx(t-h, t_0, \varphi)) &\leq v(t, x_t(t_0, \varphi)) \\ &\leq v(t_0, \varphi) < \lambda(\varepsilon_1), \quad t \geq t_0. \end{aligned} \tag{5.6}$$

We prove that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_1, \quad t \geq t_0.$$

Assume by contradiction that there exists a time instant $t_1 \geq t_0$ for which

$$\|x(t_1,t_0,\varphi) - Dx(t_1-h,t_0,\varphi)\| \geq \varepsilon_1.$$

Since

$$\begin{aligned}\|x(t_0,t_0,\varphi) - Dx(t_0-h,t_0,\varphi)\| &= \|\varphi(0) - D\varphi(-h)\| \\ &\leq (1+d\rho)\,\|\varphi\|_h < \delta_1(\varepsilon,t_0) \leq \varepsilon_1\end{aligned}$$

and $\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\|$ is a continuous function of t, there exists $t^* \in [t_0,t_1]$ such that

$$\|x(t^*,t_0,\varphi) - Dx(t^*-h,t_0,\varphi)\| = \varepsilon_1.$$

On the one hand, it follows from Eq. (5.5) that

$$v_1(x(t^*,t_0,\varphi) - Dx(t^*-h,t_0,\varphi)) \geq \lambda(\varepsilon_1).$$

On the other hand, Eq. (5.6) provides the opposite inequality

$$v_1(x(t^*,t_0,\varphi) - Dx(t^*-h,t_0,\varphi)) < \lambda(\varepsilon_1).$$

The contradiction proves that our assumption is wrong, and

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_1, \quad t \geq t_0.$$

The preceding inequality means that

$$x(t,t_0,\varphi) = Dx(t-h,t_0,\varphi) + \xi(t), \quad t \geq t_0, \tag{5.7}$$

where $\xi(t)$ is such that $\|\xi(t)\| < \varepsilon_1$, $t \geq t_0$.

For a given $t \geq t_0$ we define the entire number k such that $t \in [t_0 + (k-1)h, t_0 + kh)$. Iterating equality (5.7) $k-1$ times we obtain that

$$x(t,t_0,\varphi) = D^k x(t-kh,t_0,\varphi) + \sum_{j=0}^{k-1} D^j \xi(t-jh).$$

Since $t - kh \in [t_0 - h, t_0]$,

$$\|x(t-kh,t_0,\varphi)\| \leq \|\varphi\|_h < \delta(\varepsilon,t_0) \leq \varepsilon_1,$$

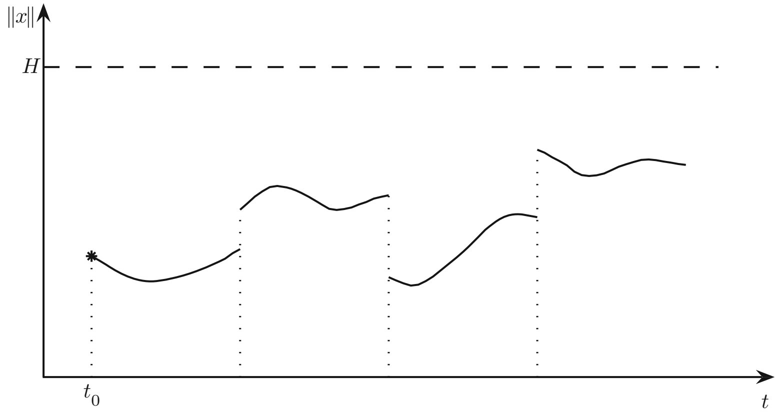

Fig. 5.1 Value of $\|x(t,t_0,\varphi\|$, the first case

and we arrive at the following inequality:

$$\begin{aligned}\|x(t,t_0,\varphi)\| &\leq \left\|D^k\right\| \|x(t-kh,t_0,\varphi)\| + \sum_{j=0}^{k-1} \left\|D^j\right\| \|\xi(t-jh)\| \\ &< d\rho^k \delta(\varepsilon,t_0) + \sum_{j=0}^{k-1} d\rho^j \varepsilon_1 < \frac{d}{1-\rho}\varepsilon_1 = \varepsilon, \quad t \geq t_0.\end{aligned}$$

This means that $\delta(\varepsilon,t_0)$ satisfies Definition 5.1, and the trivial solution of Eq. (5.1) is stable.

Necessity: Now, the trivial solution of system (5.1) is stable, and we must prove that there exists a functional $v(t,\varphi)$ that satisfies the theorem conditions.

Construction of the functional: Since the trivial solution of system (5.1) is stable, for $\varepsilon = H$ there exists $\delta(H,t_0) > 0$ such that the inequality $\|\varphi\|_h < \delta(H,t_0)$ implies that $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. We define the functional $v(t,\varphi)$ as follows:

$$v(t_0,\varphi) = \begin{cases} \sup\limits_{t\geq t_0} \|x(t,t_0,\varphi) - Dx(t-h,t_0\varphi)\|, \text{ if } \|x(t,t_0,\varphi)\| < H, \text{ for } t \geq t_0, \\ (1+d\rho)H \text{ if there exists } T \geq t_0 \text{ such that } \|x(T,t_0,\varphi)\| \geq H. \end{cases} \tag{5.8}$$

These two possibilities are illustrated in Figs. 5.1 and 5.2, respectively.

We verify first that the functional is positive definite. To this end, we must verify that it satisfies the conditions of Definition 5.4.

Condition 1: Actually, Eq. (5.8) allows us to compute $v(t_0,\varphi)$ for any $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.

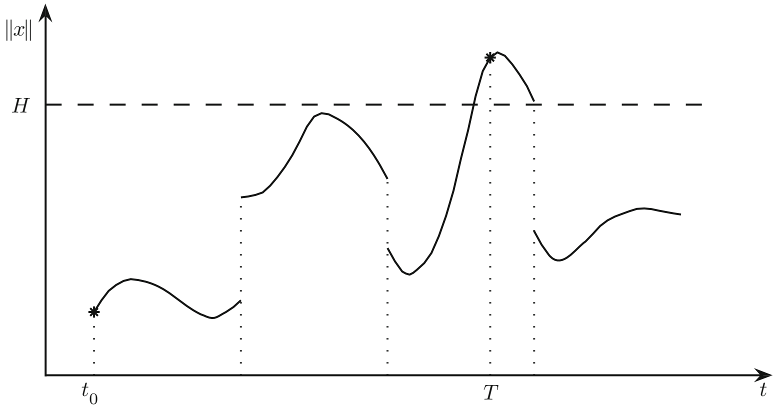

Fig. 5.2 Value of $\|x(t,t_0,\varphi\|$, the second case

Condition 2: Since for $\varphi = 0_h$ the corresponding solution is trivial, $x(t,t_0,0_h) = 0$, $t \geq t_0$, then $v(t_0,0_h) = 0$.

Condition 3: The function $v_1(x) = \|x\|$ is positive definite. In the case where $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$, we have

$$\begin{aligned} v_1(\varphi(0) - D\varphi(-h)) &= \|\varphi(0) - D\varphi(-h)\| \\ &\leq \sup_{t \geq t_0} \|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| = v(t_0,\varphi). \end{aligned}$$

And in the other case where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| \geq H$, the following inequality holds:

$$v_1(\varphi(0) - D\varphi(-h)) = \|\varphi(0) - D\varphi(-h)\| \leq (1 + d\rho)H = v(t_0,\varphi).$$

Condition 4: Given $t_0 \geq 0$, the stability of the trivial solution means that for any $\varepsilon > 0$ there exists $\delta_1 = \delta(\frac{\varepsilon}{1+d\rho}, t_0) > 0$ such that $\|\varphi\|_h < \delta_1$ implies

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon}{1 + d\rho}, \quad t \geq t_0.$$

This means that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \leq \|x(t,t_0,\varphi)\| + d\rho\, \|x(t-h,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

The preceding inequality demonstrates that

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) \leq \varepsilon.$$

This observation makes it clear that for a fixed $t_0 \geq 0$ the functional $v(t_0, \varphi)$ is continuous in φ at the point 0_h.

Now we check that functional (5.8) satisfies the theorem condition. First, we consider the case where $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. In this case, given two time instants t_1 and t_2 such that $t_2 > t_1 \geq t_0$, we have that

$$v(t_1, x_{t_1}(t_0,\varphi)) = \sup_{t \geq t_1} \|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\|$$

and

$$v(t_2, x_{t_2}(t_0,\varphi)) = \sup_{t \geq t_2} \|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\|.$$

Since for the second value the range of the supremum is smaller than that for the first one, we conclude that

$$v(t_2, x_{t_2}(t_0,\varphi)) \leq v(t_1, x_{t_1}(t_0,\varphi)).$$

This means that along the solution the functional $v(t, x_t(t_0,\varphi))$ does not increase as a function of t. In the second case, where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| \geq H$, we have the equality

$$v(t_2, x_{t_2}(t_0,\varphi)) = v(t_1, x_{t_1}(t_0,\varphi)) = (1 + d\rho)H,$$

and, once again, the functional does not increase along the solution of system (5.1).□

Remark 5.2. On the one hand, functional (5.8) has only an academic value. Obviously, we cannot use such functionals in applications. On the other hand, it demonstrates that the Lyapunov–Krasovskii approach is universal: for any system with a stable trivial solution there are positive-definite functionals satisfying Theorem 5.4.

Theorem 5.5. *The trivial solution of system (5.1) is uniformly stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions are satisfied:*

1. *The value of the functional along the solutions of the system, $v(t,x_t)$, as a function of t does not increase.*
2. *The functional is continuous in φ at the point 0_h, uniformly for $t \geq 0$.*

Proof. Sufficiency: We use notations from the proof of the sufficiency part of Theorem 5.4. Now the functional $v(t,\varphi)$ is continuous in φ at the point 0_h, uniformly for $t \geq 0$, so there exists a positive value $\delta_1(\varepsilon)$ such that the inequality $v(t_0,\varphi) < \lambda(\varepsilon_1)$ holds for any $t_0 \geq 0$ and $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h < \delta_1(\varepsilon)$. Therefore, the value

$$\delta(\varepsilon) = \frac{\delta_1(\varepsilon)}{1 + d\rho}$$

does not depend on t_0. The remainder of the sufficiency part of the proof coincides with that of Theorem 5.4.

Necessity: The uniform stability of the trivial solution of system (5.1) implies that δ can be chosen independently of t_0, $\delta = \delta(\varepsilon)$. We show that functional (5.8) satisfies the second condition of the theorem. Let us select for a given $\varepsilon > 0 (\varepsilon < H)$ the value

$$\delta_1 = \delta\left(\frac{\varepsilon}{1 + d\rho}\right).$$

Then, for any $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h < \delta_1$ and $t_0 \geq 0$, we have that

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon}{1 + d\rho}, \text{ for } t \geq t_0.$$

This means that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \leq \|x(t,t_0,\varphi)\| + d\rho\,\|x(t-h,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

The preceding inequality demonstrates that

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) \leq \varepsilon, \quad t_0 \geq 0.$$

In other words, functional (5.8) is continuous in φ at the point 0_h, uniformly with respect to $t_0 \geq 0$. □

Corollary 5.3. *Let the condition of Theorem 5.4 be fulfilled, and let the functional $v(t,\varphi)$ admit an upper estimate of the form*

$$v(t,\varphi) \leq v_2(\varphi), \quad t \geq 0,\ \varphi \in PC^1([-h,0], R^n),\ \text{with } \|\varphi\|_h \leq H,$$

with a positive-definite functional $v_2(\varphi)$; then the trivial solution of system (5.1) is uniformly stable.

Theorem 5.6. *The trivial solution of system (5.1) is asymptotically stable if and only if the following conditions hold.*

1. *There exists a positive-definite functional $v(t,\varphi)$, defined for $t \geq 0$ and $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h \leq H$ such that along the solutions of the system $v(t,x_t)$, as a function of t, does not increase.*
2. *For any $t_0 \geq 0$ there exists a positive value $\mu(t_0)$ such that if $\varphi \in PC^1([-h,0], R^n)$ and $\|\varphi\|_h < \mu(t_0)$, then $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t - t_0 \to \infty$.*

Proof. Sufficiency: Since the matrix D is Schur stable, there exists $d \geq 1$ and $\rho \in (0,1)$ such that the inequality $\|D^k\| \leq d\rho^k$ holds for $k \geq 0$. The first condition of the theorem implies the stability of the trivial solution of system (5.1); see Theorem 5.4. Thus, for any $\varepsilon \in (0,H)$ and $t_0 \geq 0$ there exists $\delta(\varepsilon,t_0) > 0$ such that if $\|\varphi\|_h < \delta(\varepsilon,t_0)$, then $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. Let us define the value

$$\Delta(\varepsilon,t_0) = \min\left\{\delta\left(\varepsilon,t_0\right),\mu\left(t_0\right)\right\}.$$

Now, given an initial function $\varphi \in PC^1\left([-h,0],R^n\right)$ such that $\|\varphi\|_h < \Delta(\varepsilon,t_0)$, we will demonstrate that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$. The functional $v(t,\varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$such that

$$v_1\left(\varphi(0) - D\varphi(-h)\right) \leq v\left(t,\varphi\right)$$

for $t \geq 0$ and $\varphi \in PC^1\left([-h,0],R^n\right)$, with $\|\varphi\|_h \leq H$. For a given $\varepsilon_1 > 0 (\varepsilon_1 < \varepsilon)$ we set

$$\varepsilon_2 = \frac{1-\rho}{2d}\varepsilon_1 > 0$$

and define the positive value

$$\alpha = \min_{\varepsilon_2 \leq \|x\| \leq \varepsilon} v_1(x).$$

By the second condition of the theorem, there exists $T > 0$ such that $v(t,x_t(t_0,\varphi)) < \alpha$ for $t \geq t_0 + T$ and

$$v_1(x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)) \leq v(t,x_t(t_0,\varphi)) < \alpha, \quad t - t_0 \geq T,$$

so we must conclude that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_2, \quad t - t_0 \geq T.$$

This means that

$$x(t,t_0,\varphi) = Dx(t-h,t_0,\varphi) + \nu(t), \quad t - t_0 \geq T, \tag{5.9}$$

where

$$\|\nu(t)\| < \varepsilon_2, \quad t - t_0 \geq T.$$

For a given $t \geq t_0 + T$ we define the integer number k such that $t \in [t_0 + T + (k-1)h, t_0 + T + kh)$. Then, iterating equality (5.9) $(k-1)$ times, we obtain that

$$x(t,t_0,\varphi) = \sum_{j=0}^{k-1} D^j \nu(t - jh) + D^k x(t - kh, t_0, \varphi)$$

and

$$\|x(t,t_0,\varphi)\| \leq \sum_{j=0}^{k-1} d\rho^j \varepsilon_2 + d\rho^k \varepsilon < \frac{d}{1-\rho}\varepsilon_2 + d\rho^k \varepsilon \leq \frac{1}{2}\varepsilon_1 + d\rho^k \varepsilon, \quad t - t_0 \geq T.$$

Since $\rho^k \to 0$ as $k \to \infty$, then, starting from some k_0, the following inequality holds:

$$d\rho^k \varepsilon < \frac{1}{2}\varepsilon_1, \quad k \geq k_0.$$

This means that $\|x(t,t_0,\varphi)\| < \varepsilon_1$ for $t \geq t_0 + T + k_0 h$, and we conclude that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$. Hence, the previously defined value $\Delta(t_0,\varepsilon)$ satisfies Definition 5.2.

Necessity: In this part of the proof we make use of functional (5.8). In the proof of Theorem 5.4 it was demonstrated that the functional is positive definite and does not increase along the solutions of system (5.1). This means that the functional satisfies the first condition of the theorem.

We address the second condition of the theorem and choose the value $\mu(t_0)$ as follows:

$$\mu(t_0) = \Delta(H,t_0) > 0.$$

Now for any initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \mu(t_0)$, we know that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$. This means that for any $\varepsilon_1 > 0$ there exists $t_1 \geq t_0$ such that

$$\|x(t,t_0,\varphi)\| < \frac{1}{1+d\rho}\varepsilon_1, \quad t \geq t_1.$$

According to Eq. (5.8), we have

$$\begin{aligned} v(t,x_t(t_0,\varphi)) &= \sup_{s\geq t}\|x(s,t_0,\varphi) - Dx(s-h,t_0,\varphi)\| \\ &\leq \frac{1}{1+d\rho}\varepsilon_1 + \frac{d\rho}{1+d\rho}\varepsilon_1 = \varepsilon_1, \quad t \geq t_1 + h. \end{aligned}$$

The preceding observation proves that $v(t,x_t(t_0,\varphi))$ tends to zero as $t - t_0 \to \infty$. □

The following statement gives sufficient conditions for the asymptotic stability of the trivial solution of system (5.1).

Theorem 5.7. *The trivial solution of system (5.1) is asymptotically stable if there exist a positive-definite functional $v(t,\varphi)$ and a positive-definite function $w(x)$ such that along the solutions of the system the functional $v(t,\varphi)$ is differentiable and its time derivative satisfies the inequality*

$$\frac{\mathrm{d}v(t,x_t)}{\mathrm{d}t} \leq -w(x(t) - Dx(t-h)).$$

Proof. Since the matrix D is Schur stable, there exists $d \geq 1$ and $\rho \in (0,1)$ such that the inequality $\left\|D^k\right\| \leq d\rho^k$ holds for $k \geq 0$.

Observe first that the theorem conditions imply that of Theorem 5.4; therefore, the trivial solution of system (5.1) is stable, i.e., for any $t_0 \geq 0$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0) > 0$, which satisfies Definition 5.1. Let us set

$$\Delta(\varepsilon, t_0) = \delta(\frac{\varepsilon}{1+d\rho}, t_0) > 0.$$

Given $t_0 \geq 0$ and an initial function $\varphi \in PC^1([-h,0], R^n)$ such that $\|\varphi\|_h < \Delta(\varepsilon, t_0)$, we have that

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon}{1+d\rho}, \quad t \geq t_0,$$

and

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0 + h. \tag{5.10}$$

First we demonstrate that

$$x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi) \to 0, \text{ as } t - t_0 \to \infty. \tag{5.11}$$

Assume by contradiction that this is not the case; then there exists $\alpha > 0$ and a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k - t_0 \to \infty$, as $k \to \infty$ such that

$$\|x(t_k,t_0,\varphi) - Dx(t_k - h,t_0,\varphi)\| \geq \alpha, \quad k \geq 1.$$

Without loss of generality we may assume that $t_{k+1} - t_k \geq h$ for $k \geq 0$. It follows from system (5.1) that

$$x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi) = [x(t_k,t_0,\varphi) - Dx(t_k - h,t_0,\varphi)] + \int_{t_k}^{t} f(s, x_s(t_0,\varphi))\mathrm{d}s, \quad t \geq t_k,$$

and since $\|x(t_k,t_0,\varphi) - Dx(t_k-h,t_0,\varphi)\| \geq \alpha$, then

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \geq \alpha - M(\varepsilon)(t - t_k), \quad t \geq t_k$$

(see condition *(i)* of Theorem 5.1). Hence, for any $k \geq 1$ the following inequality holds:

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \geq \frac{\alpha}{2}, \quad t \in [t_k, t_k + \tau],$$

where

$$\tau = \min\left\{h, \frac{\alpha}{2M(\varepsilon)}\right\}.$$

Because the function $w(x)$ is positive definite, we have that

$$\beta = \min_{\frac{\alpha}{2} \leq \|x\| \leq \varepsilon} w(x) > 0.$$

The second condition of the theorem implies that

$$\begin{aligned} v\left(t, x_t(t_0, \varphi)\right) &\leq v(t_0, \varphi) - \int_{t_0}^{t} w(x(s, t_0, \varphi) - Dx(s-h, t_0, \varphi))\mathrm{d}s \\ &\leq v(t_0, \varphi) - \tau\beta N(t), \end{aligned}$$

where $N(t)$ denotes the number of segments $[t_k, t_k + \tau]$ that belong to $[t_0, t]$. Since $N(t) \to \infty$ as $t - t_0 \to \infty$, we have that $v\left(t, x_t(t_0, \varphi)\right)$ becomes negative for sufficiently large t, which contradicts the positive definiteness of the functional. The contradiction proves Eq. (5.11). This means that

$$x(t, t_0, \varphi) = Dx(t-h, t_0, \varphi) + \xi(t), \quad t \geq t_0,$$

and $\xi(t) \to 0$ as $t - t_0 \to \infty$. Given a positive value $\varepsilon_1 < \varepsilon$, there exists $t_1 > t_0$ such that

$$\|\xi(t)\| < \frac{1-\rho}{2d}\varepsilon_1, \quad t \geq t_1.$$

Let us define k_0 such that $d\rho^k\varepsilon < \frac{1}{2}\varepsilon_1$ for $k \geq k_0$. Now for any $t \geq t_1 + k_0 h$ we have

$$x(t, t_0, \varphi) = \sum_{j=0}^{k_0-1} D^j \xi(t - jh) + D^{k_0} x(t - k_0 h, t_0, \varphi)$$

and

$$\begin{aligned} \|x(t, t_0, \varphi)\| &\leq \sum_{j=0}^{k_0-1} \left\|D^j\right\| \|\xi(t - jh)\| + \left\|D^{k_0}\right\| \|x(t - k_0 h, t_0, \varphi)\| \\ &< \frac{d}{1-\rho}\left(\frac{1-\rho}{2d}\varepsilon_1\right) + d\rho^{k_0}\varepsilon < \varepsilon_1, \end{aligned}$$

and we arrive at the conclusion that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$. This means that the previously defined positive value $\Delta(\varepsilon, t_0)$ satisfies Definition 5.2, and the trivial solution of system (5.1) is asymptotically stable. □

Now we provide a criterion of the uniform asymptotic stability of the trivial solution of system (5.1).

Theorem 5.8. *The trivial solution of system (5.1) is uniformly asymptotically stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions hold.*

1. *The functional is continuous in φ at the point 0_h, uniformly for $t \geq 0$.*
2. *There exists a positive value μ_1 such that $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t-t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$.*

Proof. Sufficiency: Comparing this theorem with Theorems 5.5 and 5.6 we conclude that the trivial solution of system (5.1) is uniformly stable and asymptotically stable. Therefore, for a given $\varepsilon > 0$ there exists

$$\Delta(\varepsilon) = \min\left\{\frac{1}{2}\delta(\varepsilon), \mu_1\right\} > 0$$

such that the following properties hold.

1. Given $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \Delta(\varepsilon)$, we have that $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$.
2. $x(t,t_0,\varphi) \to 0$, as $t-t_0 \to \infty$.

Now we define the positive value

$$H_1 = \Delta(H).$$

The functional $v(t,\varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$ such that for $t \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$, the following inequality holds:

$$v_1(\varphi(0) - D\varphi(-h)) \leq v(t,\varphi).$$

For a given $\varepsilon_1 > 0(\varepsilon_1 < \varepsilon)$ we set

$$\varepsilon_2 = \frac{1-\rho}{2d}\varepsilon_1 > 0$$

and define the positive value

$$\alpha = \min_{\varepsilon_2 \leq \|x\| \leq \varepsilon} v_1(x).$$

By the second condition of the theorem, there exists $T > 0$ such that for any $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$, the following inequality holds:

$$v(t,x_t(t_0,\varphi)) < \alpha, \; t-t_0 \geq T.$$

This implies that

$$v_1(x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)) < \alpha, \quad t - t_0 \geq T,$$

and we conclude that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_2, \quad t - t_0 \geq T,$$

for any $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$. And we again arrive at equality (5.9). Applying the arguments used in the proof of the sufficiency part of Theorem 5.6 we obtain the inequality

$$\|x(t,t_0,\varphi)\| \leq \frac{1}{2}\varepsilon_1 + d\rho^k H, \quad t - t_0 \geq T.$$

Since $\rho^k \to 0$ as $k \to \infty$, then, starting from some k_0, the following inequality holds:

$$d\rho^k H < \frac{1}{2}\varepsilon_1, \quad k \geq k_0.$$

This means that $\|x(t,t_0,\varphi)\| < \varepsilon_1$for $t - t_0 \geq \max\{T, k_0 h\}$, and we conclude that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$. Therefore, the previously defined values $\Delta(\varepsilon)$ and H_1 satisfy Definition 5.2. This concludes the proof of the sufficiency part of the theorem.

Necessity: The uniform asymptotic stability of the trivial solution of system (5.1) implies that functional (5.8) satisfies the first condition of the theorem. Set

$$\mu_1 = \frac{1}{2}\Delta(H),$$

where $\Delta(\varepsilon)$ is from Definition 5.2. Now, given $\varepsilon_1 > 0$, then for any $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$, there exists $T > 0$ such that

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon_1}{1 + d\rho}, \quad t - t_0 \geq T,$$

and

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_1, \quad t - t_0 \geq T + h.$$

This means that functional (5.8) satisfies the inequality

$$v(t, x_t(t_0,\varphi)) \leq \varepsilon_1, \quad t - t_0 \geq T + h,$$

for any $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$. In other words, under the conditions of the theorem, the value $v(t, x_t(t_0,\varphi))$ decreases monotonically to zero as $t-t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$. This concludes the proof of the necessity part. □

Theorem 5.9. *The trivial solution of system (5.1) is exponentially stable if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions are satisfied.*

1. *There are two positive constants α_1, α_2 for which the inequalities*

$$\alpha_1 \|\varphi(0) - D\varphi(-h)\|^2 \leq v(t,\varphi) \leq \alpha_2 \|\varphi\|_h^2$$

   hold for $t \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.
2. *The functional is differentiable along the solutions of the system, and there exists a positive constant σ_1 such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t,x_t) + 2\sigma_1 v(t,x_t) \leq 0.$$

Proof. Because the matrix D is Schur stable, there exist $d \geq 1$ and $\rho \in (0,1)$ such that the inequality $\left\|D^k\right\| \leq d\rho^k$ holds for $k \geq 0$. There exists $\sigma_2 > 0$ such that $\rho = \mathrm{e}^{-\sigma_2 h}$.

If we define the positive-definite function $v_1(x) = \alpha_1 \|x\|^2$ and the positive-definite functional $v_2(\varphi) = \alpha_2 \|\varphi\|_h^2$, then it becomes evident that the functional $v(t,\varphi)$ satisfies the conditions of Theorem 5.5. Therefore, the trivial solution of system (5.1) is uniformly stable. This means that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the inequality $\|\varphi\|_h < \delta(\varepsilon)$ implies $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. Let us set

$$\Delta_0 = \Delta(H).$$

We will demonstrate that this value satisfies Definition 5.3. To this end, we assume that $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, $\|\varphi\|_h < \Delta_0$. The corresponding solution $x(t,t_0,\varphi)$ is such that

$$\|x(t,t_0,\varphi)\| < H, \quad t \geq t_0.$$

The second condition of the theorem implies

$$v(t,x_t(t_0,\varphi)) \leq v(t_0,\varphi)\mathrm{e}^{-2\sigma_1(t-t_0)}, \quad t \geq t_0.$$

Applying the first condition we obtain the inequalities

$$\begin{aligned}\alpha_1 \|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\|^2 &\leq v(t_0,\varphi)\mathrm{e}^{-2\sigma_1(t-t_0)} \\ &\leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma_1(t-t_0)}, \quad t \geq t_0.\end{aligned}$$

And, finally, we arrive at the exponential estimate

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \leq \gamma_1 \|\varphi\|_h \mathrm{e}^{-\sigma_1(t-t_0)}, \quad t \geq t_0,$$

where

$$\gamma_1 = \sqrt{\frac{\alpha_2}{\alpha_1}}.$$

This means that

$$x(t,t_0,\varphi) = Dx(t-h,t_0,\varphi) + \eta(t), \quad t \geq t_0, \tag{5.12}$$

where

$$\|\eta(t)\| \leq \gamma_1 \|\varphi\|_h e^{-\sigma_1(t-t_0)}, \quad t \geq t_0.$$

For a given $t \geq t_0$ we define an integer number k such that $t \in [t_0+(k-1)h, t_0+kh)$. After $k-1$ iterations of equality (5.12) we obtain

$$x(t,t_0,\varphi) = \sum_{j=0}^{k-1} D^j \eta(t-jh) + D^k x(t-kh,t_0,\varphi).$$

The last equality implies that

$$\begin{aligned}\|x(t,t_0,\varphi)\| &\leq \sum_{j=0}^{k-1} \left\|D^j\right\| \|\eta(t-jh)\| + \left\|D^k\right\| \|\varphi\|_h \\ &\leq \sum_{j=0}^{k-1} \left(de^{-\sigma_2 jh}\right)\left(\gamma_1 \|\varphi\|_h e^{-\sigma_1(t-jh-t_0)}\right) + de^{-\sigma_2 kh} \|\varphi\|_h \\ &\leq \gamma_1 d \left(\sum_{j=0}^{k-1} e^{-\sigma_2 jh} e^{-\sigma_1(t-jh-t_0)}\right) \|\varphi\|_h + de^{-\sigma_2 kh} \|\varphi\|_h .\end{aligned}$$

If we set $\sigma_0 = \min\{\sigma_1, \sigma_2\}$, then

$$\|x(t,t_0,\varphi)\| \leq d\left[\gamma_1 k e^{-\sigma_0(t-t_0)} + e^{-\sigma_0 kh}\right] \|\varphi\|_h .$$

It follows from the definition of k that $(k-1)h \leq t-t_0 < kh$, hence

$$\begin{aligned}\|x(t,t_0,\varphi)\| &\leq d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right] e^{-\sigma_0(t-t_0)} \|\varphi\|_h \\ &= \left(d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right] e^{-\mu(t-t_0)}\right) e^{-(\sigma_0-\mu)(t-t_0)} \|\varphi\|_h ,\end{aligned}$$

where $\mu \in (0,\sigma)$. Observe that the function

$$d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right] e^{-\mu(t-t_0)} \to 0, \text{ as } t-t_0 \to \infty,$$

i.e., the function is bounded,

$$d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right]\mathrm{e}^{-\mu(t-t_0)}\leq\gamma,\quad t\geq t_0,$$

and we arrive at the exponential estimate for the solutions of system (5.1):

$$\|x(t,t_0,\varphi)\|\leq\gamma\mathrm{e}^{-\sigma(t-t_0)}\|\varphi\|_h,\quad t\geq t_0,$$

where $\gamma\geq 1$ and $\sigma=\sigma_0-\mu>0$. □

5.6 Notes and References

There are several forms in which to present neutral type time-delay systems. In this book we use the one proposed in the fundamental monograph [23]. This form assumes that solutions may have discontinuity points, but the difference $x(t)-Dx(t-h)$ remains continuous for $t\geq t_0$ (Assumption 5.1). The main reason to restrict our study to the case of systems with this simple difference operator is that a highly complicated stability analysis of more general classes of difference operators would be required. To the best of our knowledge, even in the case of several delays, stability conditions often become extremely sensitive to small variations in delays. An exhaustive stability study of more general classes of difference operators can be found in [23].

In the exposition of the existence and uniqueness theorem in Sect. 5.2 we follow an excellent source [19]. For the continuity properties of the solutions see [3,19,23].

A comprehensive treatise on the Lyapunov–Krasovskii approach to the stability analysis of neutral type time-delay systems is given in [44]. Our method of presenting basic stability results in Sect. 5.5 was inspired by [19,72].

A list of contributions with sufficient stability results, mainly presented in the form of special linear matrix inequalities, can be found in [64]; see also [58] and references therein.

Chapter 6
Linear Systems

In this chapter we consider the class of neutral type linear systems with one delay. We define the fundamental matrix of such a system and present the Cauchy formula for the solution of an initial value problem. This formula is used to compute a quadratic functional with a given time derivative along the solutions of the time-delay system. It is demonstrated that this functional is defined by a special matrix valued function, which is called a Lyapunov matrix for a time-delay system. A thorough analysis of the basic properties of the matrix is included. Complete type functionals are introduced, and various applications of the functionals are discussed.

6.1 Preliminaries

In this chapter we consider a linear neutral type time-delay system of the form

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[x(t)-Dx(t-h)\right]=A_0x(t)+A_1x(t-h),\quad t\geq 0,\tag{6.1}$$

where $h>0$ and A_0, A_1, and D are given real $n\times n$ matrices. The system is time invariant, so we assume that $t_0=0$ and

$$x(\theta)=\varphi(\theta),\quad \theta\in[-h,0].$$

We set $PC^1([-h,0],R^n)$ as the space of initial functions.

Remark 6.1. Recall our agreement (Assumptions 5.1–5.3) that the following conditions hold:

1. The difference $x(t)-Dx(t-h)$ is continuous and differentiable for $t\geq t_0$, except for possibly a countable number of points.

V.L. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*, Control Engineering, DOI 10.1007/978-0-8176-8367-2_6,

2. In (6.1) the right-hand-side derivative is assumed in the origin, $t = 0$. By default, such an agreement remains valid in those situations where only a one-sided variation of the independent variable is allowed.
3. A solution $x(t)$ satisfies system (6.1) almost everywhere.

6.1.1 Fundamental Matrix

Definition 6.1. Let the $n \times n$ matrix $K(t)$ be a solution of the matrix equation

$$\frac{\mathrm{d}}{\mathrm{dt}}[K(t) - K(t-h)D] = K(t)A_0 + K(t-h)A_1, \quad t \geq 0,$$

that satisfies the following conditions:

1. Initial condition: $K(t) = 0_{n\times n}$, for $t < 0$, and $K(0) = I$;
2. Sewing condition: $K(t) - K(t-h)D$ is continuous for $t \geq 0$.
   Then $K(t)$ is known as the fundamental matrix of system (6.1).

Remark 6.2. The fundamental matrix $K(t)$ is also a solution of the equation

$$\frac{\mathrm{d}}{\mathrm{dt}}[K(t) - DK(t-h)] = A_0K(t) + A_1K(t-h), \quad t \geq 0.$$

It follows from Definition 6.1 that the matrix $K(t)$ is piecewise continuous. In fact, the sewing condition implies that

$$\Delta K(t) = \Delta K(t-h)D, \quad t \geq 0. \tag{6.2}$$

Here $\Delta K(t) = K(t+0) - K(t-0)$. The set of discontinuity points of $K(t)$ is

$$\mathcal{T} = \{\, \nu h \mid \quad \nu = 0, 1, 2, \ldots \}\,.$$

To compute the size of the jumps at these points, one must compute the solution of the jump Eq. (6.2) with the initial condition

$$\Delta K(\theta) = 0_{n\times n}, \text{ for } \theta \in [-h, 0), \text{ and } \Delta K(0) = I.$$

Lemma 6.1. *It follows directly from Eq. (6.2) that*

$$\Delta K(t) = \begin{cases} 0_{n\times n}, & t \neq \nu h \\ D^{\nu}, & t = \nu h; \quad \nu \geq 0. \end{cases}$$

The value of the fundamental matrix $K(t)$ at a discontinuity point coincides with its right-hand-side limit at the point

$$K(\nu h) = K(\nu h + 0) = \lim_{\varepsilon \to +0} K(\nu h + \varepsilon).$$

6.1.2 Cauchy Formula

Theorem 6.1. *Given an initial function $\varphi \in PC^1([-h,0],R^n)$, the solution $x(t,\varphi)$ of system (6.1) admits the following explicit expression:*

$$x(t,\varphi) = [K(t) - K(t-h)D]\,\varphi(0) + \int_{-h}^{0} K(t-h-\theta)\left[D\varphi'(\theta) + A_1\varphi(\theta)\right] \mathrm{d}\theta, \quad t \geq 0. \tag{6.3}$$

This expression is known as the Cauchy formula for system (6.1).

Proof. Let $t > 0$ and $\xi \in (0,t)$. We compute the partial derivative

$$\begin{aligned} J &= \frac{\partial}{\partial \xi}[K(t-\xi) - K(t-\xi-h)D]x(\xi,\varphi) \\ &= -\ [K(t-\xi)A_0 - K(t-\xi-h)A_1]x(\xi,\varphi) \\ &\quad +\ [K(t-\xi) - K(t-\xi-h)D]x'(\xi,\varphi). \end{aligned}$$

Since $x(t,\varphi)$ is a solution of system (6.1), we have that

$$\begin{aligned} J_1 &= [K(t-\xi) - K(t-\xi-h)D]x'(\xi,\varphi) \\ &= K(t-\xi)\left[Dx'(\xi-h,\varphi) + A_0x(\xi,\varphi) + A_1x(\xi-h,\varphi)\right] \\ &\quad - K(t-\xi-h)Dx'(\xi,\varphi), \end{aligned}$$

and we obtain the equality

$$\begin{aligned} J &= \frac{\partial}{\partial \xi}[K(t-\xi) - K(t-\xi-h)D]x(\xi,\varphi) \\ &= K(t-\xi)A_1x(\xi-h,\varphi) - K(t-\xi-h)A_1x(\xi,\varphi) \\ &\quad + K(t-\xi)Dx'(\xi-h,\varphi) - K(t-\xi-h)Dx'(\xi,\varphi). \end{aligned}$$

Integrating the preceding equality by ξ from 0 to t we obtain that

$$\begin{aligned} x(t,\varphi) &= [K(t) - K(t-h)D]\,\varphi(0) \\ &\quad + \int_0^t K(t-\xi)A_1x(\xi-h,\varphi)\mathrm{d}\xi - \int_0^t K(t-\xi-h)A_1x(\xi,\varphi)\mathrm{d}\xi \\ &\quad + \int_0^t K(t-\xi)Dx'(\xi-h,\varphi)\mathrm{d}\xi - \int_0^t K(t-\xi-h)Dx'(\xi,\varphi)\mathrm{d}\xi \end{aligned}$$

$$= [K(t) - K(t-h)D]\,\varphi(0)$$
$$+ \int_{-h}^{0} K(t-\xi-h)A_1 x(\xi)\mathrm{d}\xi + \int_{-h}^{0} K(t-\xi-h)Dx'(\xi)\mathrm{d}\xi.$$

Because $x(\xi) = \varphi(\xi)$, $\xi \in [-h,0]$, the desired Cauchy formula is a direct consequence of the foregoing equalities. □

6.2 Lyapunov Matrices: Stable Case

Definition 6.2. System (6.1) is said to be exponentially stable if there exist $\sigma > 0$ and $\gamma \geq 1$ such that every solution of the system satisfies the inequality

$$\|x(t,\varphi)\| \leq \gamma \mathrm{e}^{-\sigma t} \|\varphi\|_h, \quad t \geq 0.$$

Here $\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|$.

Definition 6.3 ([23]). A complex number s_0 is said to be an eigenvalue of system (6.1) if it is a root of the characteristic function,

$$f(s) = \det\left(sI - s\mathrm{e}^{-sh}D - A_0 - \mathrm{e}^{-sh}A_1\right),$$

of the system. The set

$$\Lambda = \{\, s \mid f(s) = 0 \,\}$$

is known as the spectrum of the system.

The next statement shows that the property of exponential stability depends on the location of the spectrum of system (6.1).

Theorem 6.2 ([23]). *System (6.1) is exponentially stable if and only if there exists $\varepsilon > 0$ such that the spectrum of the system lies to the left of the vertical line $Re(s) = -\varepsilon$ of the complex plane,*

$$Re(s_0) < -\varepsilon, \quad s_0 \in \Lambda.$$

Remark 6.3. If system (6.1) is exponentially stable, then any eigenvalue λ_0 of the matrix D lies in the open unit disc of the complex plane, $|\lambda_0| < 1$ (Sect. 5.4).

Lemma 6.2. *Let system (6.1) be exponentially stable; then there exist $\gamma_1 \geq 1$ and $\sigma_1 > 0$ such that the following inequality holds:*

$$\|K(t)\| \leq \gamma_1 \mathrm{e}^{-\sigma_1 t}, \quad t \geq 0.$$

Proof. *Since the columns of the matrix $K(t) - K(t-h)D$ are solutions of system (6.1) with special initial conditions, we have that*

$$\|K(t) - K(t-h)D\| \leq \gamma e^{-\sigma t}, \quad t \geq 0.$$

This means that the fundamental matrix $K(t)$ satisfies the equality

$$K(t) - K(t-h)D = \Phi(t), \quad t \geq 0,$$

where $\|\Phi(t)\| \leq \gamma e^{-\sigma t}$, $t \geq 0$. Define for a given $t \geq 0$ an integer number k such that $t \in [(k-1)h, kh)$. Iterating the preceding equality $k-1$ times we obtain that

$$K(t) = K(t-kh)D^k + \sum_{j=0}^{k-1} \Phi(t-jh)D^j.$$

Because $t - kh < 0$, the first term on the right-hand side of this equality disappears. The matrix D is Schur stable, so there exist $d \geq 1$ and $\rho \in (0,1)$ for which

$$\left\|D^j\right\| \leq d\rho^j, \quad j = 0, 1, 2, \ldots$$

It is clear that $\rho = e^{-\widetilde{\sigma}h}$ for some $\widetilde{\sigma} > 0$. Now we arrive at the inequality

$$\|K(t)\| \leq \sum_{j=0}^{k-1} \|\Phi(t-jh)\| \left\|D^j\right\| \leq \gamma d \sum_{j=0}^{k-1} e^{-\sigma(t-jh)} e^{-j\widetilde{\sigma}h}.$$

If we introduce the value $\sigma_0 = \min\{\sigma, \widetilde{\sigma}\}$, then the following inequality holds:

$$\|K(t)\| \leq \gamma d k e^{-\sigma_0 t}.$$

Since $(k-1)h \leq t$,

$$k \leq \frac{t+h}{h},$$

and we obtain the inequality

$$\|K(t)\| \leq \gamma d \left(\frac{t+h}{h}\right) e^{-\sigma_0 t} = \gamma d \left(\frac{t+h}{h}\right) e^{-\mu t} e^{-(\sigma_0 - \mu)t}.$$

For any $\mu > 0$ the function

$$\gamma d \left(\frac{t+h}{h}\right) e^{-\mu t}$$

is bounded for $t \in [0, \infty)$. This means that there exists $\gamma_1 > 1$ such that

$$\gamma d \left(\frac{t+h}{h}\right) e^{-\mu t} \leq \gamma_1, \quad t \geq 0.$$

If $\mu \in (0, \sigma_0)$, then

$$\|K(t)\| \leq \gamma_1 e^{-\sigma_1 t}, \quad t \geq 0,$$

where $\sigma_1 = \sigma_0 - \mu > 0$. □

Definition 6.4. Given a symmetric matrix W, let system (6.1) be exponentially stable; then the $n \times n$ matrix

$$U(\tau) = \int_0^\infty K^T(t) W K(t+\tau) \mathrm{d}t \tag{6.4}$$

is known as a *Lyapunov matrix* of the system associated with W.

Lemma 6.2 justifies the convergence of the improper integral on the right-hand side of (6.4).

Lemma 6.3. *Let system (6.1) be exponentially stable. Then the Lyapunov matrix $U(\tau)$ is continuous for $\tau \geq 0$.*

Proof. The following inequality holds for $\tau \geq 0$:

$$\left\| K^T(t) W K(t+\tau) \right\| \leq \gamma_1^2 \|W\| \mathrm{e}^{-\sigma_1(2t+\tau)}, \quad t \geq 0.$$

For a given $\tau_0 \geq 0$ we have the difference

$$U(\tau_0+\xi) - U(\tau_0) = \int_0^\infty K^T(t) W \left[K(t+\tau_0+\xi) - K(t+\tau_0) \right] \mathrm{d}t.$$

Assume that $|\xi| < \delta$; then

$$\left\| K^T(t) W K(t+\tau_0+\xi) \right\| \leq \gamma_1^2 \|W\| \mathrm{e}^{-\sigma_1(2t+\tau_0-\delta)}, \quad t \geq 0.$$

Given $\varepsilon > 0$, there exists T such that

$$\int_T^\infty \left\| K^T(t) W K(t+\tau_0+\xi) \right\| \mathrm{d}t < \frac{1}{4}\varepsilon, \quad |\xi| < \delta.$$

This implies that

$$\begin{aligned}
\|U(\tau_0+\xi) - U(\tau_0)\| \leq & \int_0^T \left\| K^T(t) W \left[K(t+\tau_0+\xi) - K(t+\tau_0) \right] \right\| \mathrm{d}t \\
& + \int_T^\infty \left\| K^T(t) W K(t+\tau_0+\xi) \right\| \mathrm{d}t \\
& + \int_T^\infty \left\| K^T(t) W K(t+\tau_0) \right\| \mathrm{d}t \\
& < \gamma_1 \|W\| \int_0^T \|K(t+\tau_0+\xi) - K(t+\tau_0)\| \mathrm{d}t + \frac{1}{2}\varepsilon.
\end{aligned}$$

Now we estimate the integral

$$\int_0^T \|K(t+\tau_0+\xi)-K(t+\tau_0)\|\,\mathrm{d}t \leq \int_0^{T+\tau_0} \|K(s+\xi)-K(s)\|\,\mathrm{d}s.$$

The matrix $K(t)$ is continuous for $t \geq 0$ except for points of the set $\mathcal{T}$ (Sect. 6.1.1). We denote by N the smallest integer such that $Nh \geq \tau_0 + T$, and we assume that $2\delta < h$; then

$$\int_0^{T+\tau_0} \|K(s+\xi)-K(s)\|\,\mathrm{d}s \leq \sum_{j=0}^{N} \int_{jh-\delta}^{jh+\delta} \|K(s+\xi)-K(s)\|\,\mathrm{d}t$$
$$+\sum_{j=0}^{N-1} \int_{jh+\delta}^{(j+1)h-\delta} \|K(s+\xi)-K(s)\|\,\mathrm{d}s.$$

It is evident that

$$\int_{jh-\delta}^{jh+\delta} \|K(s+\xi)-K(s)\|\,\mathrm{d}t \leq 4\gamma_1\delta, \quad j=0,1,\dots,N.$$

Since the difference $K(s+\xi)-K(s)$ is continuous in s and ξ for $s \in [jh+\delta,(j+1)h-\delta]$, $|\xi| < \delta$, there exists $\varepsilon(\delta) \to 0$ as $\delta \to 0$ such that the inequalities

$$\int_{jh+\delta}^{(j+1)h-\delta} \|K(s+\xi)-K(s)\|\,\mathrm{d}t < \varepsilon(\delta), \quad j=0,1,\dots,N-1,$$

hold for any ξ, with $|\xi| < \delta$. If we select $\delta > 0$ such that

$$4\gamma_1^2\,\|W\|\,(N+1)\delta < \frac{1}{4}\varepsilon, \text{ and } \gamma_1 N\,\|W\|\,\varepsilon(\delta) < \frac{1}{4}\varepsilon,$$

then for any ξ with $|\xi| < \delta$

$$\|U(\tau_0+\xi)-U(\tau_0)\| < \varepsilon.$$

This proves the continuity property. □

Lemma 6.4. *Let system (6.1) be exponentially stable. Lyapunov matrix (6.4) satisfies the following properties.*

1. Dynamic property:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau)-U(\tau-h)D\right] = U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \geq 0; \qquad (6.5)$$

2. *Symmetry property:*

$$U(-\tau) = U^T(\tau); \tag{6.6}$$

3. *Algebraic property:*

$$\begin{aligned} -W &= A_0^T U(0) + U(0)A_0 - A_0^T U(-h)D - D^T U(h)A_0 \\ &\quad + A_1^T U(h) + U(-h)A_1 - A_1^T U(0)D - D^T U(0)A_1. \end{aligned} \tag{6.7}$$

Proof. *Dynamic property*: By definition of the Lyapunov matrix,

$$U(\tau) - U(\tau-h)D = \int_0^\infty K^T(t)W\left[K(t+\tau) - K(t+\tau-h)D\right]\mathrm{d}t, \quad \tau \geq 0.$$

It follows from Lemma 6.2 that the improper integral on the right-hand side of the preceding equality converges absolutely and uniformly with respect to $\tau \geq 0$. For $t+\tau \geq 0$ the matrix $K(t+\tau) - K(t+\tau-h)D$ is differentiable and

$$\frac{\partial}{\partial \tau}\left[K(t+\tau) - K(t+\tau-h)D\right] = K(t+\tau)A_0 + K(t+\tau-h)A_1.$$

For similar reasons the integral

$$\begin{aligned} J &= \int_0^\infty K^T(t)W\left(\frac{\partial}{\partial \tau}\left[K(t+\tau) - K(t+\tau-h)D\right]\right)\mathrm{d}t \\ &= \int_0^\infty K^T(t)W\left[K(t+\tau)A_0 + K(t+\tau-h)A_1\right]\mathrm{d}t \end{aligned}$$

converges absolutely and uniformly with respect to $\tau \geq 0$. Therefore, we conclude that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau) - U(\tau-h)D\right] &= \int_0^\infty K^T(t)W\left(\frac{\partial}{\partial \tau}\left[K(t+\tau) - K(t+\tau-h)D\right]\right)\mathrm{d}t \\ &= U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \geq 0. \end{aligned}$$

It is important to recall (Remark 6.1) that at $\tau = 0$ we understand in (6.5) the right-hand-side derivative.

Symmetry property: Equality (6.6) is a direct consequence of (6.4).

Algebraic property: To check this property, we first differentiate the product

$$R(t) = [K(t) - K(t-h)D]^T W [K(t) - K(t-h)D],$$
$$\frac{\mathrm{d}}{\mathrm{d}t}R(t) = [K(t)A_0 + K(t-h)A_1]^T W [K(t) - K(t-h)D] + [K(t) - K(t-h)D]^T W [K(t)A_0 + K(t-h)A_1],$$

and then integrate the obtained equality by t from 0 to ∞.

On the one hand, the exponential stability of system (6.1) provides that the integral of the left-hand side of the preceding equality is equal to $-W$.

On the other hand, a direct application of (6.4) shows that the integration of the right-hand side leads to the equality

$$-W = A_0^T U(0) - A_0^T U(-h)D + A_1^T U^T(-h) - A_1^T U(0)D + U(0)A_0 + U(-h)A_1 - D^T U^T(-h)A_0 - D^T U(0)A_1,$$

which after some minor transformations coincides with (6.7). □

Corollary 6.1. *It follows from Lemma 6.3 and property (6.6) that the matrix $U(\tau)$ is continuous for $\tau \leq 0$.*

Corollary 6.2. *The derivative*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[U(\tau) - U(\tau - h)D]$$

is continuous for $\tau \geq 0$.

Lemma 6.5. *Let system (6.1) be exponentially stable. The first derivative of the matrix $U(\tau)$ is continuous for $\tau \in [0,h]$.*

Proof. According to Remark 6.1, at $\tau = 0$ and $\tau = h$ we understand the right-hand-side derivative and the left-hand-side derivative, respectively.

The matrix $U(\tau)$ satisfies Eq. (6.5). Since

$$\left[-D^T U(\tau) + U(\tau - h)\right]^T = U(h-\tau) - U(-\tau)D$$

and $h - \tau \geq 0$ for $\tau \in [0,h]$, then by (6.5),

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T U(\tau) + U(\tau - h)\right]^T = -[U(h-\tau)A_0 + U(-\tau)A_1] = -\left[A_1^T U(\tau) + A_0^T U(\tau - h)\right]^T.$$

and we arrive at the equality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T U(\tau) + U(\tau - h)\right] = -A_1^T U(\tau) - A_0^T U(\tau - h), \quad \tau \in [0,h].$$

Now we multiply the preceding equality by the matrix D from the right-hand side and sum the result with (6.5):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau)-D^TU(\tau)D\right] = U(\tau)A_0+U(\tau-h)A_1 \\ -A_1^TU(\tau)D-A_0^TU(\tau-h)D, \quad \tau\in[0,h]. \tag{6.8}$$

All eigenvalues of the matrix D lie in the open unit disc (Remark 6.3). This implies that the Schur operator $S(X)=X-D^TXD$ is regular. The right-hand-side expression in (6.8) is continuous (Lemma 6.3 and Corollary 6.1). It proves the continuity property of the first derivative of the Lyapunov matrix. □

We may present now property (6.7) in an alternative form.

Lemma 6.6. *The algebraic property (6.7) of the Lyapunov matrix $U(\tau)$ can be written as*

$$-W=\Delta U'(0)-D^T\Delta U'(0)D, \tag{6.9}$$

where

$$\Delta U'(0)=U'(+0)-U'(-0)=\lim_{\tau\to+0}\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}-\lim_{\tau\to-0}\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}.$$

Proof. On the one hand, we may conclude from (6.8) that

$$U'(+0)-D^TU'(+0)D=U(0)A_0+U(-h)A_1-A_1^TU(0)D-A_0^TU(-h)D.$$

On the other hand, according to (6.6),

$$\frac{\mathrm{d}U(-\tau)}{\mathrm{d}\tau}=\left[\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right]^T,$$

so $U'(-0)=-\left[U'(+0)\right]^T$, and we obtain that

$$U'(-0)-D^TU'(-0)D=-\left[U(0)A_0+U(-h)A_1-A_1^TU(0)-A_0^TU(-h)D\right]^T.$$

Now,

$$\begin{aligned}\Delta U'(0)-D^T\Delta U'(0)D &= U(0)A_0+U(-h)A_1-A_1^TU(0)D-A_0^TU(-h)D\\ &\quad+\left[U(0)A_0+U(-h)A_1-A_1^TU(0)D-A_0^TU(-h)D\right]^T\\ &=-W.\end{aligned}$$

□

Remark 6.4. The new form of the algebraic property demonstrates that $\Delta U'(0)=P$, where P satisfies the matrix equation

$$P-D^TPD=-W. \tag{6.10}$$

6.3 Functional $v_0(\varphi)$

Assume that system (6.1) is exponentially stable. Then, for a given quadratic form x^TWx there exists a quadratic functional $v_0(\varphi)$, $\varphi \in PC^1([-h,0],R^n)$, such that along the solutions of system (6.1) the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0. \tag{6.11}$$

Equation (6.11) defines the functional $v_0(\varphi)$ up to an additive constant. We select this constant in such a way that $v_0(0_h) = 0$. Integrating the equation from $t = 0$ to $t = T > 0$ we obtain

$$v_0(x_T(\varphi)) - v_0(\varphi) = -\int_0^T x^T(t)Wx(t)\mathrm{d}t.$$

The exponential stability of Eq. (6.1) implies that $x_T(\varphi) \to 0_h$ as $T \to \infty$, and we arrive at the equality

$$v_0(\varphi) = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t, \quad \varphi \in PC^1([-h,0],R^n).$$

If we replace in the preceding equality $x(t,\varphi)$ by Cauchy formula (6.3), then

$$\begin{aligned}
v_0(\varphi) &= \varphi^T(0)\left(\int_0^\infty [K(t)-K(t-h)D]^T W [K(t)-K(t-h)D]\,\mathrm{d}t\right)\varphi(0) \\
&\quad + 2\varphi^T(0)\int_{-h}^0 \left(\int_0^\infty [K(t)-K(t-h)D]^T WK(t-h-\theta)\mathrm{d}t\right) \\
&\quad \times \left[D\varphi'(\theta)+A_1\varphi(\theta)\right]\mathrm{d}\theta \\
&\quad + \int_{-h}^0\int_{-h}^0 \left[D\varphi'(\theta_1)+A_1\varphi(\theta_1)\right]^T \left(\int_0^\infty K^T(t-h-\theta_1)WK(t-h-\theta_2)\mathrm{d}t\right) \\
&\quad \times \left[D\varphi'(\theta_2)+A_1\varphi(\theta_2)\right]\mathrm{d}\theta_2\mathrm{d}\theta_1.
\end{aligned}$$

Let $U(\tau)$ be a Lyapunov matrix associated with W; then the quadratic functional is of the form

$$\begin{aligned}
v_0(\varphi) &= \varphi^T(0)\left[U(0)-D^TU(h)-U(-h)D+D^TU(0)D\right]\varphi(0) \\
&\quad + 2\varphi^T(0)\int_{-h}^0 \left[U(-h-\theta)-D^TU(-\theta)\right]\left[D\varphi'(\theta)+A_1\varphi(\theta)\right]\mathrm{d}\theta
\end{aligned}$$

$$+\int_{-h}^{0}\left[D\varphi'(\theta_1)+A_1\varphi(\theta_1)\right]^T$$
$$\times\left(\int_{-h}^{0}U(\theta_1-\theta_2)\left[D\varphi'(\theta_2)+A_1\varphi(\theta_2)\right]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1. \qquad (6.12)$$

6.4 Lyapunov Matrices: General Case

Given a symmetric matrix W, the associated Lyapunov matrix of system (6.1) is defined as the improper integral (6.4), where $K(t)$ is the fundamental matrix of system (6.1). Certainly, this definition makes sense only for the case where system (6.1) is exponentially stable; otherwise the improper integral is not well defined. Now we provide an alternative definition of Lyapunov matrices that does not require the exponential stability of system (6.1).

Definition 6.5. Given a symmetric matrix W, a Lyapunov matrix of system (6.1) associated with W is a solution of Eq. (6.5) that satisfies properties (6.6) and (6.7).

First we check that functional (6.12) with a newly defined Lyapunov matrix $U(\tau)$ satisfies Eq. (6.11).

Theorem 6.3. *Let the matrix $U(\tau)$ in functional (6.12) satisfy Definition 6.5. Then the time derivative of the functional along the solutions of system (6.1) satisfies (6.11).*

Proof. Let $x(t)$ be a solution of system (6.1). We start with the first term of $v_0(x_t)$ [see (6.12)]:

$$R_0(t)=x^T(t)\left[U(0)-D^TU(h)-U(-h)D+D^TU(0)D\right]x(t).$$

The time derivative of the term is

$$\frac{\mathrm{d}}{\mathrm{d}t}R_0(t)=2x^T(t)\left[U(0)-D^TU(h)-U(-h)D+D^TU(0)D\right]x'(t).$$

Consider the term

$$\begin{aligned}R_1(t)&=2x^T(t)\int_{-h}^{0}\left[U(-h-\theta)-D^TU(-\theta)\right]\left[Dx'(t+\theta)+A_1x(t+\theta)\right]\mathrm{d}\theta\\&=2x^T(t)\int_{t-h}^{t}\left[U(h+s-t)-U(s-t)D\right]^T\left[Dx'(s)+A_1x(s)\right]\mathrm{d}s.\end{aligned}$$

For this term

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_1(t) = {} & 2\left[x'(t)\right]^T \int_{t-h}^{t} \left[U(h+s-t) - U(s-t)D\right]^T \left[Dx'(s) + A_1x(s)\right] \mathrm{d}s \\ & + 2x^T(t)\left[U(h) - U(0)D\right]^T \left[Dx'(t) + A_1x(t)\right] \\ & - 2x^T(t)\left[U(0) - U(-h)D\right]^T \left[Dx'(t-h) + A_1x(t-h)\right] \\ & + 2x^T(t) \int_{t-h}^{t} \left(\frac{\partial}{\partial t}\left[U(h+s-t) - U(s-t)D\right]\right)^T \left[Dx'(s) + A_1x(s)\right] \mathrm{d}s,\end{aligned}$$

or, if we return to the original integration variables and use the fact that the matrix $U(\tau)$ satisfies dynamic property (6.5),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_1(t) = {} & 2x^T(t)\left[U(-h)D - D^TU(0)D\right]x'(t) \\ & + x^T(t)\left[U(-h)A_1 + A_1^TU(h) - D^TU(0)A_1 - A_1^TU(0)D\right]x(t) \\ & - 2x^T(t)\left[U(0) - U(-h)D\right]^T \left[Dx'(t-h) + A_1x(t-h)\right] \\ & + 2\left[x'(t)\right]^T \int_{-h}^{0} \left[U(h+\theta) - U(\theta)D\right]^T \left[Dx'(t+\theta) + A_1x(t+\theta)\right] \mathrm{d}\theta \\ & - 2x^T(t) \int_{-h}^{0} \left[U(h+\theta)A_0 + U(\theta)A_1\right]^T \left[Dx'(t+\theta) + A_1x(t+\theta)\right] \mathrm{d}\theta.\end{aligned}$$

We now address the term

$$\begin{aligned}R_2(t) = {} & \int_{-h}^{0} \left[Dx'(t+\theta_1) + A_1x(t+\theta_1)\right]^T \\ & \times \left(\int_{-h}^{0} U(\theta_1 - \theta_2)\left[Dx'(t+\theta_2) + A_1x(t+\theta_2)\right] \mathrm{d}\theta_2\right) \mathrm{d}\theta_1 \\ = {} & \int_{t-h}^{t} \left[Dx'(s_1) + A_1x(s_1)\right]^T \left(\int_{t-h}^{t} U(s_1 - s_2)\left[Dx'(s_2) + A_1x(s_2)\right] \mathrm{d}s_2\right) \mathrm{d}s_1.\end{aligned}$$

The time derivative of this term is the following

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) &= \left[Dx'(t)+A_1x(t)\right]^T \int_{t-h}^{t} U(t-s_2)\left[Dx'(s_2)+A_1x(s_2)\right]\mathrm{d}s_2 \\ &\quad - \left[Dx'(t-h)+A_1x(t-h)\right]^T \int_{t-h}^{t} U(t-h-s_2)\left[Dx'(s_2)+A_1x(s_2)\right]\mathrm{d}s_2 \\ &\quad + \left(\int_{t-h}^{t}\left[Dx'(s_1)+A_1x(s_1)\right]^T U(s_1-t)\mathrm{d}s_1\right)\left[Dx'(t)+A_1x(t)\right] \\ &\quad - \left(\int_{t-h}^{t}\left[Dx'(s_1)+A_1x(s_1)\right]^T U(s_1-t+h)\mathrm{d}s_1\right) \\ &\quad \times \left[Dx'(t-h)+A_1x(t-h)\right].\end{aligned}$$

And applying the symmetry property (6.6) we arrive at the following expression:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) &= 2\left[Dx'(t)+A_1x(t)\right]^T \int_{-h}^{0}\left[U(\theta)\right]^T\left[Dx'(t+\theta)+A_1x(t+\theta)\right]\mathrm{d}\theta \\ &\quad - 2\left[Dx'(t-h)+A_1x(t-h)\right]^T \\ &\quad \times \int_{-h}^{0}\left[U(h+\theta)\right]^T\left[Dx'(t+\theta)+A_1x(t+\theta)\right]\mathrm{d}\theta.\end{aligned}$$

Now, collecting the computed derivatives we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) &= -x^T(t)\left[U(-h)A_1 - D^TU(0)A_1 + A_1^TU(h) - A_1^TU(0)D\right]x(t) \\ &\quad + 2x^T(t)\left[U(0)-U(-h)D\right]^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]-A_1x(t-h)\right) \\ &\quad + 2\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]-A_0x(t)-A_1x(t-h)\right)^T \\ &\quad \times \int_{-h}^{0}\left[U(h+\theta)\right]^T\left[Dx'(t+\theta)+A_1x(t+\theta)\right]\mathrm{d}\theta.\end{aligned}$$

The last term on the right-hand side of the preceding equality is equal to zero since $x(t)$ is a solution of system (6.1). By the same reason we have the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] - A_1 x(t-h) = A_0 x(t).$$

This means that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) &= x^T(t)\left[U(-h)A_1 - D^T U(0)A_1 + A_1^T U(h) - A_1^T U(0)D\right]x(t)\\ &\quad + x^T(t)\left[A_0^T U(0) - A_0^T U(-h)D + U(0)A_0 - D^T U(h)A_0\right]x(t).\end{aligned}$$

According to algebraic property (6.7), the right-hand side of the preceding equality is equal to $-x^T(t)Wx(t)$, and we arrive at the final conclusion that

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0. \qquad \square$$

Now we check that Definition 6.5 is consistent with Definition 6.4.

Theorem 6.4. *Let system (6.1) be exponentially stable. Then matrix (6.4) is a unique solution of Eq. (6.5), which satisfies properties (6.6) and (6.7).*

Proof. The fact that matrix (6.4) is a continuous solution of Eq. (6.5) and satisfies properties (6.6) and (6.7) was proven in Lemmas 6.3 and 6.4.

We show now that this solution is unique. Assume by contradiction that for a given symmetric matrix W there are two such solutions, $U_j(\tau)$, $j = 1,2$. Then we define two functionals, $v_0^{(j)}(\varphi)$, $j = 1,2$, of the form (6.12), the first one with $U(\tau) = U_1(\tau)$ and the other with $U(\tau) = U_2(\tau)$. Both $U_1(\tau)$ and $U_2(\tau)$ satisfy (6.5)–(6.7), so by Theorem 6.3,

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0^{(j)}(x_t) = -x^T(t)Wx(t), \quad t \geq 0; \quad \text{for } j = 1,2.$$

This observation implies that the difference $\Delta v_0(x_t) = v_0^{(2)}(x_t) - v_0^{(1)}(x_t)$ is such that

$$\Delta v_0(x_t(\varphi)) = \Delta v_0(\varphi), \quad t \geq 0.$$

System (6.1) is exponentially stable, so $x_t(\varphi) \to 0_h$ as $t \to \infty$, and we conclude that $0 = \Delta v_0(\varphi)$ for any initial function $\varphi \in PC^1([-h,0],R^n)$. We write this equality in the explicit form

$$\begin{aligned}0 &= \varphi^T(0)\left[\Delta U(0) - D^T\Delta U(h) - \Delta U(-h)D + D^T\Delta U(0)D\right]\varphi(0)\\ &\quad + 2\varphi^T(0)\int_{-h}^{0}\left[\Delta U(h+\theta) - \Delta U(\theta)D\right]^T\left[D\varphi'(\theta) + A_1\varphi(\theta)\right]\mathrm{d}\theta\\ &\quad + \int_{-h}^{0}\left[D\varphi'(\theta_1) + A_1\varphi(\theta_1)\right]^T\left(\int_{-h}^{0}\Delta U(\theta_1 - \theta_2)\left[D\varphi'(\theta_2) + A_1\varphi(\theta_2)\right]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1.\end{aligned} \tag{6.13}$$

Here the matrix $\Delta U(\tau) = U_2(\tau) - U_1(\tau)$.

By definition, the matrix $\Delta U(\tau)$ is a solution of Eq. (6.5), which satisfies properties (6.6) and (6.7) with $W = 0_{n\times n}$.

Consider the following initial function:

$$\varphi(\theta) = \begin{cases} \gamma, \text{ for } \theta \in [-\varepsilon, 0] \\ 0, \text{ for } \theta \in [-h, -\varepsilon), \end{cases} \tag{6.14}$$

where γ is a constant vector and ε is a sufficiently small positive value. For this function condition (6.13) has the form

$$\begin{aligned} 0 = {} & \gamma^T \left[\Delta U(0) - D^T \Delta U(h) - \Delta U(-h) D + D^T \Delta U(0) D\right] \gamma \\ & + 2\gamma^T \left[\Delta U(h-\varepsilon) - \Delta U(-\varepsilon) D\right]^T D\gamma \\ & + 2\varepsilon\gamma^T \left[\Delta U(h) - \Delta U(0) D\right]^T A_1 \gamma + \gamma^T \left[D^T \Delta U(0) D\right] \gamma \\ & + 2\varepsilon\gamma^T \left[\Delta U(0) D\right]^T A_1 \gamma + o(\varepsilon), \end{aligned}$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$, as $\varepsilon \to +0$. The preceding equality may be written as

$$0 = \alpha_0 + \varepsilon\alpha_1 + o(\varepsilon),$$

from which we deduce that both coefficients α_0 and α_1 should be equal to zero. The first coefficient

$$\begin{aligned} \alpha_0 = {} & \gamma^T \left[\Delta U(0) - D^T \Delta U(h) - \Delta U(-h) D + D^T \Delta U(0) D\right] \gamma \\ & + 2\gamma^T \left[\Delta U(h) - \Delta U(0) D\right]^T D\gamma + \gamma^T \left[D^T \Delta U(0) D\right] \gamma \\ = {} & \gamma^T \Delta U(0) \gamma. \end{aligned}$$

Because γ is an arbitrary vector and matrix $\Delta U(0)$ is symmetric, we conclude that

$$\Delta U(0) = 0_{n\times n}. \tag{6.15}$$

By Lemma 6.6, the matrices $U_j(\tau)$, $j = 1, 2$, satisfy (6.9). This means that the matrix $\Delta U(\tau)$ satisfies the equality

$$\left[\Delta U'(+0) - \Delta U'(-0)\right] - D^T \left[\Delta U'(+0) - \Delta U'(-0)\right] D = 0_{n\times n}.$$

Since the matrix D is Schur stable (Remark 6.3), the last equality implies that

$$\Delta U'(+0) - \Delta U'(-0) = 0_{n\times n}. \tag{6.16}$$

The same result follows from the equality $\alpha_1 = 0$.

Now we consider an initial function of the form

$$\varphi(\theta) = \begin{cases} \gamma, \text{ for } \theta \in [-\varepsilon, 0], \\ \mu, \text{ for } \theta \in [-\tau_0, -\tau_0 + \varepsilon], \\ 0, \text{ for all other points of } [-h, 0]. \end{cases} \tag{6.17}$$

It is assumed here that γ and μ are two constant vectors, $\tau_0 \in (0, h)$, and $\varepsilon > 0$ is such that $-\tau_0 + 2\varepsilon < 0$. Substituting this function into (6.13) and taking into account (6.15) we observe that the first term

$$\begin{aligned} R_1 &= \varphi^T(0)\left[\Delta U(0) - D^T \Delta U(h) - \Delta U(-h)D + D^T \Delta U(0)D\right]\varphi(0) \\ &= -\,\gamma^T\left[D^T \Delta U(h) + \Delta U(-h)D\right]\gamma. \end{aligned}$$

The term

$$\begin{aligned} R_2 &= 2\varphi^T(0)\int_{-h}^{0}\left[\Delta U(h+\theta) - \Delta U(\theta)D\right]^T\left[D\varphi'(\theta) + A_1\varphi(\theta)\right]\mathrm{d}\theta \\ &= 2\gamma^T\left[\Delta U(h-\tau_0) - \Delta U(-\tau_0)D\right]^T D\mu \\ &\quad -2\gamma^T\left[\Delta U(h-\tau_0+\varepsilon) - \Delta U(-\tau_0+\varepsilon)D\right]^T D\mu \\ &\quad +2\gamma^T\left[\Delta U(h-\varepsilon) - \Delta U(-\varepsilon)D\right]^T D\gamma \\ &\quad +2\gamma^T\left(\int_{-\tau_0}^{-\tau_0+\varepsilon}\left[\Delta U(h+\theta) - \Delta U(\theta)D\right]^T\mathrm{d}\theta\right)A_1\mu \\ &\quad +2\gamma^T\left(\int_{-\varepsilon}^{0}\left[\Delta U(h+\theta) - \Delta U(\theta)D\right]^T\mathrm{d}\theta\right)A_1\gamma. \end{aligned}$$

For sufficiently small ε we have

$$\begin{aligned} R_2 &= 2\gamma^T\left[\Delta U(h) - \Delta U(0)D\right]^T D\gamma - 2\varepsilon\gamma^T\left[\Delta U'(h-\tau_0) - \Delta U'(-\tau_0)D\right]^T D\mu \\ &\quad -2\varepsilon\gamma^T\left[\Delta U'(h-0) - \Delta U'(-0)D\right]^T D\gamma \\ &\quad +2\varepsilon\gamma^T\left[\Delta U(h-\tau_0) - \Delta U(-\tau_0)D\right]^T A_1\mu \\ &\quad +2\varepsilon\gamma^T\left[\Delta U(h) - \Delta U(0)D\right]^T A_1\gamma + o(\varepsilon). \end{aligned}$$

We split the last term into three summands as follows:

$$R_3 = \int_{-h}^{0} \left[D\varphi'(\theta_1) + A_1\varphi(\theta_1)\right]^T \left(\int_{-h}^{0} \Delta U(\theta_1 - \theta_2) \left[D\varphi'(\theta_2) + A_1\varphi(\theta_2)\right] \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$= \int_{-h}^{0} \left[D\varphi'(\theta_1)\right]^T \left(\int_{-h}^{0} \Delta U(\theta_1 - \theta_2) D\varphi'(\theta_2) \mathrm{d}\theta_2 \right) d\theta_1$$

$$+ 2\int_{-h}^{0} \left[D\varphi'(\theta_1)\right]^T \left(\int_{-h}^{0} \Delta U(\theta_1 - \theta_2) A_1\varphi(\theta_2) \mathrm{d}\theta_2 \right) d\theta_1$$

$$+ \int_{-h}^{0} [A_1\varphi(\theta_1)]^T \left(\int_{-h}^{0} \Delta U(\theta_1 - \theta_2) A_1\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1.$$

We start with the term

$$R_{31} = \int_{-h}^{0} \left[D\varphi'(\theta_1)\right]^T \left(\int_{-h}^{0} \Delta U(\theta_1 - \theta_2) D\varphi'(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$= \mu^T D^T \left[\Delta U(0) D\mu - \Delta U(-\varepsilon) D\mu + \Delta U(-\tau_0 + \varepsilon) D\gamma\right]$$

$$- \mu^T D^T \left[\Delta U(\varepsilon) D\mu - \Delta U(0) D\mu + \Delta U(-\tau_0 + 2\varepsilon) D\gamma\right]$$

$$+ \gamma^T D^T \left[\Delta U(\tau_0 - \varepsilon) D\mu - \Delta U(\tau_0 - 2\varepsilon) D\mu + \Delta U(0) D\gamma\right]$$

$$= \gamma^T D^T \Delta U(0) D\gamma - \varepsilon \mu^T D^T \left[\Delta U'(+0) - \Delta U'(-0)\right] D\mu$$

$$- 2\varepsilon \mu^T D^T \Delta U'(-\tau_0) D\gamma + o(\varepsilon).$$

Then we evaluate the term

$$R_{32} = 2\int_{-h}^{0} \left[D\varphi'(\theta_1)\right]^T \left(\int_{-h}^{0} \Delta U(\theta_1 - \theta_2) A_1\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$= 2 \int_{-\tau_0}^{-\tau_0+\varepsilon} \left[\mu^T D^T \Delta U(-\tau_0 - \theta)\right.$$

$$\left. - \mu^T D^T \Delta U(-\tau_0 + \varepsilon - \theta) + \gamma^T D^T \Delta U(-\varepsilon - \theta)\right] A_1 \mu \mathrm{d}\theta$$

$$
\begin{aligned}
&+2\int_{-\varepsilon}^{0}\left[\mu^T D^T \Delta U(-\tau_0-\theta)\right.\\
&\left.-\mu^T D^T \Delta U(-\tau_0+\varepsilon-\theta)+\gamma^T D^T \Delta U(-\varepsilon-\theta)\right] A_1\gamma \mathrm{d}\theta\\
&= 2\varepsilon\gamma^T D^T \Delta U(\tau_0)A_1\mu + o(\varepsilon).
\end{aligned}
$$

Finally, we observe that the term

$$
R_{33}=\int_{-h}^{0}\left[A_1\varphi(\theta_1)\right]^T\left(\int_{-h}^{0}\Delta U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 = o(\varepsilon).
$$

Taking into account equalities (6.15) and (6.16) we obtain that

$$
\begin{aligned}
R_1+R_2+R_3 = &-2\varepsilon\gamma^T\left[\Delta U'(h-\tau_0)-\Delta U'(-\tau_0)D\right]^T D\mu\\
&-2\varepsilon\gamma^T\left[\Delta U'(h-0)-\Delta U'(-0)D\right]^T D\gamma\\
&+2\varepsilon\gamma^T\left[\Delta U(h-\tau_0)-\Delta U(-\tau_0)D\right]^T A_1\mu\\
&+2\varepsilon\gamma^T\left[\Delta U(h)-\Delta U(0)D\right]^T A_1\gamma - 2\varepsilon\mu^T D^T\Delta U'(-\tau_0)D\gamma\\
&+2\varepsilon\gamma^T D^T\Delta U(\tau_0)A_1\mu + o(\varepsilon).
\end{aligned}
$$

Since for $\mu=0$ function (6.17) coincides with (6.14), the sum of the quadratic terms with respect to γ in $I_1+I_2+I_3$ disappears. As a result,

$$
\begin{aligned}
R_1+R_2+R_3 &= -2\varepsilon\gamma^T\left[\Delta U'(h-\tau_0)\right]^T D\mu + 2\varepsilon\gamma^T\left[\Delta U(h-\tau_0)\right]^T A_1\mu + o(\varepsilon)\\
&= 2\varepsilon\gamma^T\left[\Delta U'(\tau_0-h)D+\Delta U(\tau_0-h)A_1\right]\mu + o(\varepsilon).
\end{aligned}
$$

It follows from this analysis that for function (6.17) equality (6.13) takes the form

$$
2\varepsilon\gamma^T\left[\Delta U'(\tau_0-h)D+\Delta U(\tau_0-h)A_1\right]\mu + o(\varepsilon) = 0.
$$

The preceding equality holds for arbitrarily small positive ε, so

$$
\gamma^T\left[\Delta U'(\tau_0-h)D+\Delta U(\tau_0-h)A_1\right]\mu = 0
$$

for any given vectors γ and μ. This implies that

$$
\Delta U'(\tau_0-h)D+\Delta U(\tau_0-h)A_1 = 0_{n\times n}.
$$

Recall that τ_0 is an arbitrary value from $(0,h)$, so

$$
\Delta U'(\tau-h)D+\Delta U(\tau-h)A_1 = 0_{n\times n}, \quad \tau\in(0,h).
$$

From this equality and Lemmas 6.3–6.5 we conclude that Eq. (6.5) for the matrix $\Delta U(\tau)$ takes the form

$$\Delta U'(\tau) = \Delta U(\tau)A_0, \quad \tau \in [0,h].$$

Condition (6.15) implies that $\Delta U(\tau) = 0_{n\times n}$, $\tau \in [0,h]$, or

$$U_2(\tau) = U_1(\tau), \quad \tau \in [0,h]. \qquad \square$$

6.5 Existence and Uniqueness of Lyapunov Matrices

With this new definition of the Lyapunov matrices we do not need to assume that system (6.1) is exponentially stable, but it is important to know the conditions under which the matrices do (or do not) exist. In this section we study the existence issue, as well as the uniqueness one.

For a given Lyapunov matrix $U(\tau)$ we define two auxiliary matrices

$$Y(\tau) = U(\tau), \quad Z(\tau) = U(\tau - h), \quad \tau \in [0,h]. \tag{6.18}$$

Lemma 6.7. *Let $U(\tau)$ be a Lyapunov matrix associated with W; then auxiliary matrices (6.18) satisfy the following delay-free system of matrix equations:*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[Y(\tau) - Z(\tau)D\right] = Y(\tau)A_0 + Z(\tau)A_1, \\ \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T Y(\tau) + Z(\tau)\right] = -A_1^T Y(\tau) - A_0^T Z(\tau), \end{cases} \tag{6.19}$$

and the boundary value conditions

$$\begin{cases} Y(0) = Z(h), \\ \\ -W = Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h) \\ \qquad -D^T\left[Y(h)A_0 + Z(h)A_1\right] - \left[A_0^T Z(0) + A_1^T Y(0)\right]D. \end{cases} \tag{6.20}$$

Proof. The first equation in (6.19) is a direct consequence of (6.5) and (6.18). To derive the second equation, we observe that $Z(\tau) = U^T(h-\tau)$, $\tau \in [0,h]$, so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T Y(\tau) + Z(\tau)\right] &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(h-\tau) - U((h-\tau)-h)D\right]^T \\ &= -\left[U(h-\tau)A_0 + U(-\tau)A_1\right]^T \\ &= -A_1^T Y(\tau) - A_0^T Z(\tau). \end{aligned}$$

The first boundary value condition follows directly from (6.18). The second one is the algebraic property (6.7) written in the terms of the auxiliary matrices. □

Theorem 6.5. *Given a symmetric matrix W, if a pair $(Y(\tau),Z(\tau))$ satisfies (6.19) and (6.20), then*

$$U(\tau) = \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right], \qquad \tau \in [0,h], \tag{6.21}$$

is a Lyapunov matrix associated with W if we extend it to $[-h,0)$ by setting $U(-\tau) = U^T(\tau)$ for $\tau \in (0,h]$.

Proof. Symmetry property: By definition, the matrix $U(\tau)$ satisfies symmetry property (6.6) for $\tau \in (0,h]$, and we only need to check the property for $\tau = 0$. To do this, we observe that the first equality in (6.20) implies that

$$U(0) = \frac{1}{2}\left[Y(0) + Z^T(h)\right] = \frac{1}{2}\left[Y(0) + Y^T(0)\right] = U^T(0).$$

Algebraic property: It is easy to see that the following equalities hold:

$$U(0) = \frac{1}{2}\left[Y(0) + Z^T(h)\right] = \frac{1}{2}\left[Z(h) + Y^T(0)\right]$$

and

$$U(h) = \frac{1}{2}\left[Y(h) + Z^T(0)\right], \quad U(-h) = \frac{1}{2}\left[Y^T(h) + Z(0)\right].$$

With these equalities in mind we find that

$$U(0)A_0 + U(-h)A_1 = \frac{1}{2}\left[Y(0) + Z^T(h)\right]A_0 + \frac{1}{2}\left[Y^T(h) + Z(0)\right]A_1$$

and

$$A_0^T U(0) + A_1^T U(h) = \frac{1}{2}A_0^T\left[Z(h) + Y^T(0)\right] + \frac{1}{2}A_1^T\left[Y(h) + Z^T(0)\right].$$

Therefore,

$$\begin{aligned} J_1 &= U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) \\ &= \frac{1}{2}\left[Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h)\right] \\ &\quad + \frac{1}{2}\left[Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h)\right]^T. \end{aligned}$$

In a similar way we obtain that

$$\begin{aligned} U(h)A_0 + U(0)A_1 &= \frac{1}{2}\left[Y(h) + Z^T(0)\right]A_0 + \frac{1}{2}\left[Z(h) + Y^T(0)\right]A_1 \\ &= \frac{1}{2}\left[Y(h)A_0 + Z(h)A_1\right] + \frac{1}{2}\left[A_0^T Z(0) + A_1^T Y(0)\right]^T \end{aligned}$$

and

$$\begin{aligned} A_0^T U(-h) + A_1^T U(0) &= \frac{1}{2}A_0^T\left[Y^T(h) + Z(0)\right] + \frac{1}{2}A_1^T\left[Y(0) + Z^T(h)\right] \\ &= \frac{1}{2}\left[Y(h)A_0 + Z(h)A_1\right]^T + \frac{1}{2}\left[A_0^T Z(0) + A_1^T Y(0)\right], \end{aligned}$$

so

$$\begin{aligned} J_2 &= -D^T\left[U(h)A_0 + U(0)A_1\right] - \left[A_0^T U(-h) + A_1^T U(0)\right]D \\ &= -\frac{1}{2}\left(D^T\left[Y(h)A_0 + Z(h)A_1\right] + \left[A_0^T Z(0) + A_1^T Y(0)\right]D\right) \\ &\quad -\frac{1}{2}\left(D^T\left[Y(h)A_0 + Z(h)A_1\right] + \left[A_0^T Z(0) + A_1^T Y(0)\right]D\right)^T. \end{aligned}$$

We arrive at the following equality:

$$\begin{aligned} J_1 + J_2 &= U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) \\ &\quad -D^T\left[U(h)A_0 + U(0)A_1\right] - \left[A_0^T U(-h) + A_1^T U(0)\right]D \\ &= \frac{1}{2}\left\{Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h)\right. \\ &\quad \left. - D^T\left[Y(h)A_0 + Z(h)A_1\right] - \left[A_0^T Z(0) + A_1^T Y(0)\right]D\right\} \\ &\quad + \frac{1}{2}\left\{Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h)\right. \\ &\quad \left. - D^T\left[Y(h)A_0 + Z(h)A_1\right] - \left[A_0^T Z(0) + A_1^T Y(0)\right]D\right\}^T \\ &= -W. \end{aligned}$$

This ends the proof of algebraic property (6.7).

Dynamic property: Observe first that for $\tau \in [0,h]$

$$U(\tau - h) = U^T(h-\tau) = \frac{1}{2}\left[Y^T(h-\tau) + Z(\tau)\right].$$

Finally, we compute the derivative

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau)-U(\tau-h)D\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\left[Y(\tau)+Z^T(h-\tau)\right]-\frac{1}{2}\left[Y^T(h-\tau)+Z(\tau)\right]D\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\left[Y(\tau)-Z(\tau)D\right]+\frac{1}{2}\left[-D^TY(h-\tau)+Z(h-\tau)\right]^T\right) \\
&= \frac{1}{2}\left[Y(\tau)A_0+Z(\tau)A_1\right]-\frac{1}{2}\left[-A_1^TY(h-\tau)-A_0^TZ(h-\tau)\right]^T \\
&= \frac{1}{2}\left[Y(\tau)+Z^T(h-\tau)\right]A_0+\frac{1}{2}\left[Y^T(h-\tau)+Z(\tau)\right]A_1 \\
&= U(\tau)A_0+U(\tau-h)A_1, \quad \tau\in[0,h].
\end{aligned}
$$

This proves dynamic property (6.5).

Hence, according to Definition 6.5, the matrix $U(\tau)$ is a Lyapunov matrix associated with W. □

Theorem 6.5 raises the existence and uniqueness issues for boundary value problem (6.19)–(6.20). We will address them, but first we introduce the following definition.

Definition 6.6. We say that system (6.1) satisfies the Lyapunov condition if there exists $\varepsilon>0$ such that the sum of any two eigenvalues, s_1,s_2, of the system has a module greater than ε,

$$|s_1+s_2|>\varepsilon.$$

Lemma 6.8. *System (6.1) satisfies the Lyapunov condition if and only if the following two conditions hold:*

1. *System (6.1) has no eigenvalue s_0 such that $-s_0$ is also an eigenvalue of the system.*
2. *The matrix D has no eigenvalue λ_0 such that λ_0^{-1} is also an eigenvalue of the matrix.*

Proof. Necessity: Because system (6.1) satisfies the Lyapunov condition, the first condition of the lemma obviously holds.

We now address the second condition. Let $\widetilde{\lambda}$ be a nonzero eigenvalue of the matrix D. Then there exists a complex number $\widetilde{s}$ such that $\widetilde{\lambda}=\mathrm{e}^{-h\widetilde{s}}$. It is well known – see, for example, [3] – that $\widetilde{\lambda}$ generates a neutral type chain of eigenvalues of system (6.1) of the form

$$s_k=\widetilde{s}+i\frac{2k\pi}{h}+\xi_k, \quad k=\pm1,\pm2,\dots,$$

where $\xi_k \to 0$ as $|k| \to \infty$. Here i is the imaginary unit. Assume that the second condition fails and there exists an eigenvalue λ_0 of the matrix D such that λ_0^{-1} is also an eigenvalue of the matrix. Then there exists a complex number s_0 such that $\lambda_0 = \mathrm{e}^{-hs_0}$ and $\lambda_0^{-1} = \mathrm{e}^{hs_0}$. For any given $\varepsilon > 0$ there is a sufficiently large integer N such that system (6.1) has an eigenvalue

$$s^{(1)} = s_0 + i\frac{2N\pi}{h} + \xi^{(1)},$$

where $\left|\xi^{(1)}\right| < \frac{\varepsilon}{2}$, and an eigenvalue

$$s^{(2)} = -s_0 - i\frac{2N\pi}{h} + \xi^{(2)},$$

where $\left|\xi^{(2)}\right| < \frac{\varepsilon}{2}$. As a consequence

$$\left|s^{(1)} + s^{(2)}\right| \leq \left|\xi^{(1)}\right| + \left|\xi^{(2)}\right| < \varepsilon.$$

This contradicts the Lyapunov condition. The contradiction proves the second condition of the lemma.

Sufficiency: For any $R > 0$ at most a finite set of eigenvalues of system (6.1) lies in the disc

$$K(R) = \{\ s \mid |s| \leq R\ \}.$$

When R is sufficiently large, the eigenvalues of a system with magnitude greater than R are distributed among a finite number of asymptotic chains of eigenvalues. Some of the chains are of the retarded type. The real part of eigenvalues in such a chain tends to $-\infty$ as the magnitude of the eigenvalues tends to ∞. The other chains are of the neutral type. They are generated by nonzero eigenvalues of the matrix D. Assume by contradiction that there exists a sequence $\left\{\left(s_1^{(k)}, s_2^{(k)}\right)\right\}_{k=1}^{\infty}$ of eigenvalue pairs such that

$$s_1^{(k)} + s_2^{(k)} \to 0, \text{ as } k \to \infty.$$

The first condition of the lemma implies that for any given $R > 0$ there exists $\varepsilon > 0$ such that for the eigenvalues from the disc $K(R)$ the following inequality holds:

$$|s_1 + s_2| > \varepsilon.$$

This implies that $\left|s_1^{(k)}\right| \to \infty$ and $\left|s_2^{(k)}\right| \to \infty$ as $k \to \infty$. Since

$$Re\left(s_1^{(k)} + s_2^{(k)}\right) \to 0, \text{ as } k \to \infty,$$

the eigenvalues of the sequence belong to neutral type chains of eigenvalues. The chains are generated by nonzero eigenvalues, λ_1, λ_2, of the matrix D. Since

$$\lambda_1 = \lim_{k\to\infty} \mathrm{e}^{hs_1^{(k)}}, \quad \lambda_2 = \lim_{k\to\infty} \mathrm{e}^{hs_2^{(k)}},$$

we have that

$$\lambda_1\lambda_2 = \lim_{k\to\infty} \mathrm{e}^{h\left(s_1^{(k)}+s_2^{(k)}\right)} = 1.$$

This contradicts the second condition of the lemma. The contradiction ends the proof of the sufficiency part. □

Remark 6.5. Let system (6.1) satisfy the Lyapunov condition. Then delay-free system (6.19) is regular and can be written as

$$\begin{cases} Y'(\tau) - Z'(\tau)D = Y(\tau)A_0 + Z(\tau)A_1 \\ -D^T Y'(\tau) + Z'(\tau) = -A_1^T Y(\tau) - A_0^T Z(\tau), \end{cases}$$

where

$$Y'(\tau) = \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} \text{ and } Z'(\tau) = \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}.$$

Proof. Let us add to the first equation of system (6.19) the second one multiplied from the right-hand side by the matrix D; then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[Y(\tau) - D^T Y(\tau) D\right] = Y(\tau)A_0 + Z(\tau)A_1 - A_1^T Y(\tau)D - A_0^T Z(\tau)D. \tag{6.22}$$

Because the matrix D has no eigenvalue λ_0 such that λ_0^{-1} is also an eigenvalue of the matrix, the Schur operator $S(X) = X - D^T X D$ is regular. This means that the preceding equation defines the first derivative of the matrix $Y(\tau)$. Now, if we add to the second equation of system (6.19) the first one multiplied from the left-hand side by the matrix D^T, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[Z(\tau) - D^T Z(\tau) D\right] = -A_1^T Y(\tau) - A_0^T Z(\tau) + D^T Y(\tau)A_0 + D^T Z(\tau)A_1, \tag{6.23}$$

and for the same reason this equation defines the first derivative of the matrix $Z(\tau)$.□

Remark 6.6. Let system (6.1) satisfy the Lyapunov condition. Then the second boundary value condition in (6.20) can be presented in the form

$$\left[Y'(0) - Z'(h)\right] - D^T\left[Y'(0) - Z'(h)\right] D = -W,$$

where

$$Y'(0) = \left.\frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0}, \text{ and } Z'(h) = \left.\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}\right|_{\tau=h-0}.$$

Proof. It follows from Eqs. (6.22) and (6.23) that

$$Y'(0) - D^T Y'(0) D = Y(0)A_0 + Z(0)A_1 - A_1^T Y(0)D - A_0^T Z(0)D$$

and

$$Z'(h) - D^T Z'(h) D = -A_1^T Y(h) - A_0^T Z(h) + D^T Y(h) A_0 + D^T Z(h) A_1.$$

Therefore,

$$Y'(0) - D^T Y'(0) D + Z'(h) - D^T Z'(h) D = -W. \qquad \square$$

The following auxiliary result will be needed in the proof of Theorem 6.6.

Lemma 6.9. *Let system (6.1) satisfy the Lyapunov condition. If system (6.19) admits a solution $(Y(\tau), Z(\tau))$ that satisfies the boundary value conditions (6.20) with $W = 0_{n\times n}$, then*

$$Y(\tau) = Z(h+\tau), \quad \tau \in R. \tag{6.24}$$

Proof. We check first that the matrices $Y(\tau)$ and $Z(\tau)$ satisfy the second-order delay-free matrix differential equation

$$\begin{aligned} \frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} - D^T \frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} D = {} & \frac{\mathrm{d}X}{\mathrm{d}\tau} A_0 - A_0^T \frac{\mathrm{d}X}{\mathrm{d}\tau} + D^T \frac{\mathrm{d}X}{\mathrm{d}\tau} A_1 \\ & - A_1^T \frac{\mathrm{d}X}{\mathrm{d}\tau} D + A_0^T X A_0 - A_1^T X A_1. \end{aligned} \tag{6.25}$$

To this end, we differentiate Eq. (6.22)

$$\frac{\mathrm{d}^2Y(\tau)}{\mathrm{d}\tau^2} - D^T \frac{\mathrm{d}^2Y(\tau)}{\mathrm{d}\tau^2} D = \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} A_0 + \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau} A_1 - A_1^T \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} D - A_0^T \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau} D.$$

There are two terms on the right-hand side of the last equality that depend on $\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}$. The second equation of system (6.19) implies that the first of the terms can be expressed as

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau} A_1 = D^T \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} A_1 - A_1^T Y(\tau) A_1 - A_0^T Z(\tau) A_1,$$

whereas the first equation of the system makes it possible to present the second term in the form

$$-A_0^T \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau} D = -A_0^T \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} + A_0^T Y(\tau) A_0 + A_0^T Z(\tau) A_1.$$

Substituting these expressions we arrive at the conclusion that $Y(\tau)$ satisfies Eq. (6.25). Similar manipulations prove that the matrix $Z(\tau)$ is also a solution of the equation.

Note that the Lyapunov condition implies that the matrix D has no eigenvalue λ_0 such that λ_0^{-1} is also an eigenvalue of the matrix (Lemma 6.8). This means that the Schur operator, $S(X) = X - D^T XD$, is nonsingular. Therefore, Eq. (6.25) is regular, and any solution of the equation is uniquely determined by its initial conditions $X(0), X'(0)$. According to Remark 6.6, the second condition in (6.20) can be written as follows:

$$\left[Y'(0) - Z'(h)\right] - D^T \left[Y'(0) - Z'(h)\right] D = 0_{n\times n}.$$

Since the Schur operator $S(X)$ is nonsingular, the preceding equality implies that

$$Y'(0) - Z'(h) = 0_{n\times n}.$$

If we add the first condition from Eq. (6.20), $Y(0) = Z(h)$, then identity (6.24) becomes evident. □

Corollary 6.3. *Let system (6.1) satisfy the Lyapunov condition. Then the Lyapunov matrix $U(\tau)$ associated with W satisfies the second-order delay-free matrix equation*

$$\begin{aligned}\frac{\mathrm{d}^2X(\tau)}{\mathrm{d}\tau^2} - D^T\frac{\mathrm{d}^2X(\tau)}{\mathrm{d}\tau^2}D = {} & \frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau}A_0 - A_0^T\frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau} + D^T\frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau}A_1 \\ & - A_1^T\frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau}D + A_0^TX(\tau)A_0 \\ & - A_1^TX(\tau)A_1, \qquad \tau\in[0,h], \end{aligned} \tag{6.26}$$

and the boundary value conditions

$$X'(0) - \left[X'(h)\right]^T D = X(0)A_0 + X^T(h)A_1$$

and

$$-W = \left[X'(0) + \left(X'(0)\right)^T\right] - D^T\left[X'(0) + \left(X'(0)\right)^T\right]D.$$

Theorem 6.6. *System (6.1) admits a unique Lyapunov matrix associated with a given symmetric matrix W if and only if the system satisfies the Lyapunov condition.*

Proof. Sufficiency: Given a symmetric matrix W, according to Theorem 6.5, we can compute a Lyapunov matrix associated with W if there exists a solution of the boundary value problem (6.19)–(6.20). In what follows, we demonstrate that under the Lyapunov condition the boundary value problem admits a unique solution for any symmetric matrix W.

System (6.19) is linear and time invariant. To define a particular solution of the system, one must know the initial matrices $Y_0 = Y(0)$ and $Z_0 = Z(0)$. The initial matrices have $2n^2$ unknown components. Conditions (6.20) provide a system of $2n^2$

scalar linear algebraic equations in $2n^2$ unknown components of the initial matrices. This algebraic system admits a unique solution for any symmetric matrix W if and only if the unique solution of the system with $W = 0_{n\times n}$ is a trivial one. Assume by contradiction that there exists a nontrivial solution, (Y_0, Z_0), of the algebraic system with $W = 0_{n\times n}$. These initial matrices generate a nontrivial solution, $(Y(\tau), Z(\tau))$, of boundary value problem (6.19)–(6.20) with $W = 0_{n\times n}$. By Lemma 6.9, the matrices $Y(\tau)$ and $Z(\tau)$ satisfy identity (6.24). The nontrivial solution can be presented as a sum of the eigenmotions of system (6.19):

$$(Y(\tau), Z(\tau)) = \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left(P_\nu(\tau), Q_\nu(\tau)\right).$$

Here s_ν, $\nu = 0, 1, \ldots, N$, are distinct eigenvalues of system (6.19) and $P_\nu(\tau)$ and $Q_\nu(\tau)$ are polynomials with matrix coefficients. At least one of the polynomials $P_\nu(\tau)$, say $P_0(\tau)$, is nontrivial because otherwise $Y(\tau) \equiv 0_{n\times n}$, and identity (6.24) implies that $Z(\tau) \equiv 0_{n\times n}$. Let polynomial $P_0(\tau)$ be of degree ℓ

$$P_0(\tau) = \sum_{j=0}^{\ell} \tau^j B_j,$$

where B_j, $j = 0, 1, \ldots \ell$, are $n \times n$ constant matrices and $B_\ell \neq 0_{n\times n}$. It follows from (6.24) that

$$P_0(\tau) = \mathrm{e}^{s_0 h} Q_0(\tau + h).$$

Hence $Q_0(\tau)$ is also a nontrivial polynomial of degree ℓ

$$Q_0(\tau) = \sum_{j=0}^{\ell} \tau^j C_j$$

and $C_\ell = \mathrm{e}^{-s_0 h} B_\ell \neq 0_{n\times n}$.

By (6.24) the first matrix equation in (6.19) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[Y(\tau) - Y(\tau - h)D\right] = Y(\tau)A_0 + Y(\tau - h)A_1,$$

and we obtain that

$$\begin{aligned} 0_{n\times n} = {} & \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left[s_\nu P_\nu(\tau) + \frac{\mathrm{d}P_\nu(\tau)}{\mathrm{d}\tau} - s_\nu \mathrm{e}^{-s_\nu h} P_\nu(\tau - h)D - \mathrm{e}^{-s_\nu h} \frac{\mathrm{d}P_\nu(\tau - h)}{\mathrm{d}\tau} D \right] \\ & - \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left[P_\nu(\tau)A_0 + \mathrm{e}^{-s_\nu h} P_\nu(\tau - h)A_1 \right]. \end{aligned}$$

Since all eigenvalues s_ν, $\nu = 0,1,\dots,N$, are distinct, for each ν we have the polynomial identity

$$\begin{aligned}0_{n\times n} &= s_\nu P_\nu(\tau) + \frac{\mathrm{d}P_\nu(\tau)}{\mathrm{d}\tau} - s_\nu \mathrm{e}^{-s_\nu h} P_\nu(\tau-h)D - \mathrm{e}^{-s_\nu h}\frac{\mathrm{d}P_\nu(\tau-h)}{\mathrm{d}\tau}D \\ &\quad - P_\nu(\tau)A_0 - \mathrm{e}^{-s_\nu h}P_\nu(\tau-h)A_1, \quad \nu = 0,1,\dots,N.\end{aligned}$$

If in the polynomial identity for $\nu = 0$ we collect the terms of the highest degree ℓ, then we arrive at the matrix equality

$$B_\ell\left(s_0 I - s_0\mathrm{e}^{-s_0 h}D - A_0 - \mathrm{e}^{-s_0 h}A_1\right) = 0_{n\times n}.$$

Because $B_\ell \neq 0_{n\times n}$, the preceding equality holds only if

$$\det\left(s_0 I - s_0\mathrm{e}^{-s_0 h}D - A_0 - \mathrm{e}^{-s_0 h}A_1\right) = 0,$$

and we conclude that s_0 is an eigenvalue of the original system (6.1).

Once again, identity (6.24) make it possible to present the second equation of system (6.19) as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T Z(\tau+h) + Z(h)\right] = -A_1^T Z(\tau+h) - A_0^T Z(\tau).$$

And we arrive at the new set of polynomial equalities

$$\begin{aligned}0_{n\times n} &= -s_\nu \mathrm{e}^{s_\nu h}D^T Q_\nu(\tau+h) - \mathrm{e}^{s_\nu h}D^T\frac{\mathrm{d}Q_\nu(\tau+h)}{\mathrm{d}\tau} + s_\nu Q_\nu(\tau) + \frac{\mathrm{d}Q_\nu(\tau)}{\mathrm{d}\tau} \\ &\quad + A_1^T Q_\nu(\tau+h) + A_0^T Q_\nu(\tau), \quad \nu = 0,1,\dots,N.\end{aligned}$$

If in the equality for $\nu = 0$ we collect the terms of the highest degree ℓ, then

$$-\left[(-s_0)I - (-s_0)\mathrm{e}^{-(-s_0)h}D - A_0 - \mathrm{e}^{-(-s_0)h}A_1\right]^T C_\ell = 0_{n\times n}.$$

Because $C_\ell \neq 0_{n\times n}$, the preceding equality holds only if

$$\det\left[(-s_0)I - (-s_0)\mathrm{e}^{-(-s_0)h}D - A_0 - \mathrm{e}^{-(-s_0)h}A_1\right] = 0,$$

and we conclude that $-s_0$ is also an eigenvalue of the original system (6.1). This means that system (6.1) does not satisfy the Lyapunov condition. But this contradicts the theorem condition. The contradiction proves that the only solution of boundary value problem (6.19)–(6.20), with $W = 0_{n\times n}$, is the trivial one. As was mentioned previously, this implies that for any symmetric W boundary value problem (6.19)–(6.20) admits a unique solution, and this solution generates a unique Lyapunov matrix associated with W.

Necessity: Suppose that system (6.1) does not satisfy the Lyapunov condition. By Lemma 6.8, this means that either the spectrum of the system contains a point s_0, such that $-s_0$ also belongs to the spectrum, or there is an eigenvalue λ_0 of the matrix D, such that λ_0^{-1} is also an eigenvalue of the matrix.

In the first case, there exist nontrivial vectors $\gamma, \mu \in C^n$ such that

$$\mu^T \left[s_0\left(I - \mathrm{e}^{-s_0 h} D\right) - A_0 - \mathrm{e}^{-s_0 h} A_1\right] = 0,$$

$$\left[-s_0\left(I - \mathrm{e}^{s_0 h} D\right) - A_0 - \mathrm{e}^{s_0 h} A_1\right]^T \gamma = 0.$$

We demonstrate now that there exists a nontrivial solution $(Y(\tau), Z(\tau))$ of boundary value problem (6.19)–(6.20), with $W = 0_{n\times n}$. To see this, we set $Y(\tau) = \mathrm{e}^{s_0\tau}\gamma\mu^T$ and $Z(\tau) = \mathrm{e}^{s_0(\tau-h)}\gamma\mu^T$; then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}[Y(\tau) - Z(\tau)D] &= \mathrm{e}^{s_0\tau}\gamma\mu^T s_0\left(I - \mathrm{e}^{-s_0 h}D\right) = \mathrm{e}^{s_0\tau}\gamma\mu^T\left(A_0 + \mathrm{e}^{-s_0 h}A_1\right) \\ &= Y(\tau)A_0 + Z(\tau)A_1\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T Y(\tau) + Z(\tau)\right] &= s_0\left(I - \mathrm{e}^{s_0 h}D\right)^T \mathrm{e}^{s_0(\tau-h)}\gamma\mu^T \\ &= (-A_0^T - \mathrm{e}^{s_0 h}A_1^T)\mathrm{e}^{s_0(\tau-h)}\gamma\mu^T \\ &= -A_1^T Y(\tau) - A_0^T Z(\tau).\end{aligned}$$

It is evident that $Y(\tau) = Z(\tau + h)$, so

$$Y(0) = Z(h), \text{ and } Y'(0) - Z'(h) = \left.\frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau}\right|_{\tau=0} - \left.\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}\right|_{\tau=h} = 0_{n\times n}.$$

This implies that the matrices $Y(\tau) = \mathrm{e}^{s_0\tau}\gamma\mu^T$ and $Z(\tau) = \mathrm{e}^{s_0(\tau-h)}\gamma\mu^T$ satisfy boundary value conditions (6.20), with $W = 0_{n\times n}$.

In the second case, when the matrix D has an eigenvalue λ_0 such that λ_0^{-1} is also an eigenvalue of the matrix, the Schur operator $S(X) = X - D^T XD$ is singular. Therefore, for some symmetric matrix W, Eq. (6.9) is inconsistent. □

Corollary 6.4. *Let system (6.1) satisfy the Lyapunov condition. If $(Y(\tau), Z(\tau))$ is the unique solution of boundary value problem (6.19)–(6.20) with a symmetric matrix W, then the Lyapunov matrix $U(\tau)$ associated with W is such that*

$$U(\tau) = Y(\tau), \quad \tau \in [0, h].$$

Proof. We prove first that if the pair $(Y(\tau), Z(\tau))$ is a solution of the boundary value problem, then the pair

$$\left(\widetilde{Y}(\tau), \widetilde{Z}(\tau)\right) = \left(Z^T(h-\tau), Y^T(h-\tau)\right)$$

is also a solution of the problem. To this end, we observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\widetilde{Y}(\tau) - \widetilde{Z}(\tau)D\right] &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[Z(h-\tau) - D^T Y(h-\tau)\right]^T \\ &= -\left[-A_1^T Y(h-\tau) - A_0^T Z(h-\tau)\right]^T \\ &= \widetilde{Y}(\tau)A_0 + \widetilde{Z}(\tau)A_1\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T\widetilde{Y}(\tau) + \widetilde{Z}(\tau)\right] &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[Y(h-\tau) - Z(h-\tau)D\right]^T \\ &= -\left[Y(h-\tau)A_0 + Z(h-\tau)A_1\right]^T \\ &= -A_1^T\widetilde{Y}(\tau) - A_0^T\widetilde{Z}(\tau).\end{aligned}$$

Then we have

$$\widetilde{Y}(0) - \widetilde{Z}(h) = \left[Z(h) - Y(0)\right]^T = 0_{n\times n}$$

and

$$\begin{aligned}\left[\widetilde{Y}'(0) - \widetilde{Z}'(h)\right] - D^T\left[\widetilde{Y}'(0) - \widetilde{Z}'(h)\right]D &= \left\{\left[Y'(0) - Z'(h)\right]\right. \\ &\qquad \left. -D^T\left[Y'(0) - Z'(h)\right]D\right\}^T \\ &= -W^T = -W.\end{aligned}$$

Since $(Y(\tau), Z(\tau))$ is the unique solution of boundary value problem (6.19)–(6.20), we have that

$$\widetilde{Y}(\tau) = Y(\tau), \quad \tau \in [0,h],$$

and matrix (6.21) can be written as

$$U(\tau) = \frac{1}{2}\left[Y(\tau) + \widetilde{Y}(\tau)\right] = Y(\tau), \quad \tau \in [0,h]. \qquad \square$$

Theorem 6.7. *Let system (6.1) not satisfy the Lyapunov condition. Then there exists a symmetric matrix W for which Eq.(6.5) has no solution satisfying properties (6.6) and (6.7).*

Proof. By Lemma 6.8, the theorem condition means that either there exists an eigenvalue s_0 of system (6.1), such that $-s_0$ is also an eigenvalue of the system, or there exists an eigenvalue λ_0 of matrix D, such that λ_0^{-1} is also an eigenvalue of the matrix.

In the first case system (6.1) admits two solutions of the form

$$x^{(1)}(t) = \mathrm{e}^{s_0 t}\gamma, \quad x^{(2)}(t) = \mathrm{e}^{-s_0 t}\mu,$$

where γ, μ, are nontrivial vectors.

Recall that by Lemma 2.9 there exists a symmetric matrix $\widetilde{W}$ such that

$$\gamma^T \widetilde{W} \mu \neq 0.$$

Let us set in (6.7) $W = \widetilde{W}$. Assume that Eq. (6.5) admits a solution, $\widetilde{U}(\tau)$, that satisfies properties (6.6) and (6.7).

Now we define the bilinear functional

$$\begin{aligned} z(\varphi,\psi) &= \varphi^T(0)\left[\widetilde{U}(0) - D^T\widetilde{U}(h) - \widetilde{U}(-h)D + D^T\widetilde{U}(0)D\right]\psi(0) \\ &\quad + \varphi^T(0)\int_{-h}^{0}\left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]^T\left[D\dot{\psi}(\theta) + A_1\psi(\theta)\right]\mathrm{d}\theta \\ &\quad + \int_{-h}^{0}\left[D\varphi'(\theta) + A_1\varphi(\theta)\right]^T\left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]\mathrm{d}\theta\,\psi(0) \\ &\quad + \int_{-h}^{0}\left[D\varphi'(\theta_1) + A_1\varphi(\theta_1)\right]^T\left(\int_{-h}^{0}\widetilde{U}(\theta_1-\theta_2)\left[D\dot{\psi}(\theta_2) + A_1\psi(\theta_2)\right]\mathrm{d}\theta_1\right)\mathrm{d}\theta_2. \end{aligned}$$

Here $\varphi,\psi \in PC^1([-h,0],R^n)$. Let $x(t)$ and $y(t)$ be two solutions of system (6.1). We compute the time derivative of the function

$$\begin{aligned} z(x_t,y_t) &= x^T(t)\left[\widetilde{U}(0) - D^T\widetilde{U}(h) - \widetilde{U}(-h)D + D^T\widetilde{U}(0)D\right]y(t) \\ &\quad + x^T(t)\int_{-h}^{0}\left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]^T\left[Dy'(t+\theta) + A_1y(t+\theta)\right]\mathrm{d}\theta \\ &\quad + \left(\int_{-h}^{0}\left[Dx'(t+\theta) + A_1x(t+\theta)\right]^T\left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]\mathrm{d}\theta\right)y(t) \\ &\quad + \int_{-h}^{0}\left[Dx'(t+\theta_1) + A_1x(t+\theta_1)\right]^T \\ &\quad \times\left(\int_{-h}^{0}\widetilde{U}(\theta_1-\theta_2)\left[Dy'(t+\theta_2) + A_1y(t+\theta_2)\right]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1. \end{aligned}$$

The time derivative of the first term is equal to

$$\frac{\mathrm{d}R_1(t)}{\mathrm{d}t} = \left[x'(t)\right]^T \left[\widetilde{U}(0) - D^T\widetilde{U}(h) - \widetilde{U}(-h)D + D^T\widetilde{U}(0)D\right] y(t)$$
$$+x^T(t) \left[\widetilde{U}(0) - D^T\widetilde{U}(h) - \widetilde{U}(-h)D + D^T\widetilde{U}(0)D\right] y'(t).$$

The time derivative of the second term is

$$\frac{\mathrm{d}R_2(t)}{dt} = \left[x'(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]^T \left[Dy'(t+\theta) + A_1 y(t+\theta)\right] \mathrm{d}\theta$$
$$+ x^T(t) \left[\widetilde{U}(h) - \widetilde{U}(0)D\right]^T \left[Dy'(t) + A_1 y(t)\right]$$
$$- x^T(t) \left[\widetilde{U}(0) - \widetilde{U}(-h)D\right]^T \left[Dy'(t-h) + A_1 y(t-h)\right]$$
$$- x^T(t) \int_{-h}^{0} \left[\widetilde{U}(h+\theta)A_0 + \widetilde{U}(\theta)A_1\right]^T \left[Dy'(t+\theta) + A_1 y(t+\theta)\right] \mathrm{d}\theta.$$

Now the time derivative of the third term has the form

$$\frac{\mathrm{d}R_3(t)}{\mathrm{d}t} = \left[y'(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta$$
$$+ y^T(t) \left[\widetilde{U}(h) - \widetilde{U}(0)D_1\right]^T \left[Dx'(t) + A_1 x(t)\right]$$
$$- y^T(t) \left[\widetilde{U}(0) - \widetilde{U}(-h)D\right]^T \left[Dx'(t-h) + A_1 x(t-h)\right]$$
$$- y^T(t) \int_{-h}^{0} \left[\widetilde{U}(h+\theta)A_0 + \widetilde{U}(\theta)A_1\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta.$$

Finally, the time derivative of the last term is equal to

$$\frac{\mathrm{d}R_4(t)}{\mathrm{d}t} = \left[Dx'(t) + A_1 x(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(\theta)\right]^T \left[Dy'(t+\theta) + A_1 y(t+\theta)\right] \mathrm{d}\theta$$
$$- \left[Dx'(t-h) + A_1 x(t-h)\right]^T$$
$$\times \int_{-h}^{0} \left[\widetilde{U}(\theta+h)\right]^T \left[Dy'(t+\theta) + A_1 y(t+\theta)\right] \mathrm{d}\theta$$
$$+ \left[Dy'(t) + A_1 y(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(\theta)\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta$$
$$- \left[Dy'(t-h) + A_1 y(t-h)\right]^T$$
$$\times \int_{-h}^{0} \left[\widetilde{U}(\theta+h)\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta.$$

We replace in the derivatives the terms

$$Dx'(t-h)+A_1x(t-h), \text{ and } Dy'(t-h)+A_1y(t-h)$$

by

$$x'(t)-A_0x(t), \text{ and } y'(t)-A_0y(t),$$

respectively. Then we collect all terms that have no integral factor, and the sum of such terms is equal to

$$\begin{aligned} S_1(t) = {} & \left[x'(t)\right]^T \left[\widetilde{U}(0)-D^T\widetilde{U}(h)-\widetilde{U}(-h)D+D^T\widetilde{U}(0)D\right] y(t) \\ & + x^T(t)\left[\widetilde{U}(0)-D^T\widetilde{U}(h)-\widetilde{U}(-h)D+D^T\widetilde{U}(0)D\right] y'(t) \\ & + x^T(t)\left[\widetilde{U}(-h)-D^T\widetilde{U}(0)\right]\left[Dy'(t)+A_1y(t)\right] \\ & - x^T(t)\left[\widetilde{U}(0)-D^T\widetilde{U}(h)\right]\left[y'(t)-A_0y(t)\right] \\ & + y^T(t)\left[\widetilde{U}(-h)-D^T\widetilde{U}(0)\right]\left[Dx'(t)+A_1x(t)\right] \\ & - y^T(t)\left[\widetilde{U}(0)-D^T\widetilde{U}(h)\right]\left[x'(t)-A_0x(t)\right]. \end{aligned}$$

After evident reductions the sum takes the form

$$\begin{aligned} S_1(t) = x^T(t)\Big\{ & \widetilde{U}(-h)A_1 - D^T\widetilde{U}(0)A_1 + \widetilde{U}(0)A_0 - D^T\widetilde{U}(h)A_0 \\ & + A_1^T\widetilde{U}(h) - A_1^T\widetilde{U}(0)D + A_0^T\widetilde{U}(0) - A_0^T\widetilde{U}(-h)D\Big\} y(t). \end{aligned}$$

By (6.7) this sum is equal to $-x^T(t)Wy(t)$.

Now we collect all terms that contain an integral factor and start with $x^T(t)$ or $[x'(t)]^T$. Their sum is

$$\begin{aligned} S_2(t) = {} & \left[x'(t)\right]^T \int_{-h}^{0}\left[\widetilde{U}(h+\theta)-\widetilde{U}(\theta)D\right]^T\left[Dy'(t+\theta)+A_1y(t+\theta)\right]\mathrm{d}\theta \\ & - x^T(t)\int_{-h}^{0}\left[\widetilde{U}(h+\theta)A_0+\widetilde{U}(\theta)A_1\right]^T\left[Dy'(t+\theta)+A_1y(t+\theta)\right]\mathrm{d}\theta \\ & + \left[Dx'(t)+A_1x(t)\right]^T\int_{-h}^{0}\left[\widetilde{U}(\theta)\right]^T\left[Dy'(t+\theta)+A_1y(t+\theta)\right]\mathrm{d}\theta \\ & - \left[x'(t)-A_0x(t)\right]^T\int_{-h}^{0}\left[\widetilde{U}(\theta+h)\right]^T\left[Dy'(t+\theta)+A_1y(t+\theta)\right]\mathrm{d}\theta. \end{aligned}$$

It is obvious that the sum is equal to zero.

The terms that contain an integral factor and start with $y(t)$ or $y'(t)$ are as follows:

$$
\begin{aligned}
S_3(t) = {} & \left[y'(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta \\
& - y^T(t) \int_{-h}^{0} \left[\widetilde{U}(h+\theta)A_0 + \widetilde{U}(\theta)A_1\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta \\
& + \left[Dy'(t) + A_1 y(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(\theta)\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta \\
& - \left[y'(t) - A_0 y(t)\right]^T \int_{-h}^{0} \left[\widetilde{U}(\theta+h)\right]^T \left[Dx'(t+\theta) + A_1 x(t+\theta)\right] \mathrm{d}\theta .
\end{aligned}
$$

This sum is also equal to zero. And we arrive at the final conclusion that the time derivative of the bivariate functional is equal to $-x^T(t)Wy(t)$.

On the one hand,

$$
\frac{\mathrm{d}}{\mathrm{d}t} z(x_t^{(1)}, x_t^{(2)}) = -\mathrm{e}^{(s_0 - s_0)t} \gamma^T \widetilde{W} \mu = -\gamma^T \widetilde{W} \mu \neq 0. \tag{6.27}
$$

On the other hand, a direct substitution shows that

$$
z(x_t^{(1)}, x_t^{(2)}) = -\mathrm{e}^{(s_0 - s_0)t} \gamma^T Q \mu = \gamma^T Q \mu ,
$$

where the matrix

$$
\begin{aligned}
Q = {} & \widetilde{U}(0) - D^T \widetilde{U}(h) - \widetilde{U}(-h)D + D^T \widetilde{U}(0)D \\
& + \int_{-h}^{0} \left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right]^T \left[-s_0 D + A_1\right] \mathrm{e}^{-s_0\theta} \mathrm{d}\theta \\
& + \int_{-h}^{0} \left[s_0 D + A_1\right]^T \left[\widetilde{U}(h+\theta) - \widetilde{U}(\theta)D\right] \mathrm{e}^{s_0\theta} \mathrm{d}\theta \\
& + \int_{-h}^{0} \left[s_0 D + A_1\right]^T \left(\int_{-h}^{0} \widetilde{U}(\theta_1 - \theta_2) \left[-s_0 D + A_1\right] \mathrm{e}^{s_0(\theta_1 - \theta_2)} \mathrm{d}\theta_1 \right) \mathrm{d}\theta_2
\end{aligned}
$$

does not depend on t, so

$$
\frac{\mathrm{d}}{\mathrm{d}t} z(x_t^{(1)}, x_t^{(2)}) = \frac{\mathrm{d}}{\mathrm{d}t} \gamma^T Q \mu = 0.
$$

The last equality contradicts inequality (6.27); therefore, our assumption that Eq. (6.5) admits a solution that satisfies properties (6.6) and (6.7), with $W = \widetilde{W}$, is wrong.

In the second case, when the matrix D has an eigenvalue λ_0 such that λ_0^{-1} is also an eigenvalue of the matrix, for some symmetric matrices W Eq. (6.9) is inconsistent, so there are no Lyapunov matrices associated with such W, either. □

6.6 Computational Issue

It is important to have an efficient numerical procedure for the computation of Lyapunov matrices. In this section we present such a procedure.

We have already seen that if boundary value problem (6.19)–(6.20) has a solution, then, by Theorem 6.5, this solution generates a Lyapunov matrix associated with a given W. In the case where the boundary value problem admits a unique solution, $(Y(\tau), Z(\tau))$, we have

$$U(\tau) = Y(\tau), \quad \tau \in [0,h].$$

According to Theorem 6.6 the Lyapunov condition (Definition 6.6) guarantees the existence of a unique solution of boundary value problem (6.19)–(6.20) with an arbitrary symmetric matrix W. In the rest of this section we assume that system (6.1) satisfies this condition.

In vector form, system (6.19) is written

$$\begin{pmatrix} I\otimes I & -I\otimes D \\ -D^T\otimes I & I\otimes I \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = \begin{pmatrix} I\otimes A_0 & I\otimes A_1 \\ -A_1^T\otimes I & -A_0^T\otimes I \end{pmatrix}\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix}. \tag{6.28}$$

Here $y(\tau) = vec(Y(\tau))$ and $z(\tau) = vec(Z(\tau))$. Since under the Lyapunov condition system (6.19) is regular (Remark 6.5), system (6.28) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = L\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix},$$

where

$$L = \begin{pmatrix} I\otimes I & -I\otimes D \\ -D^T\otimes I & I\otimes I \end{pmatrix}^{-1}\begin{pmatrix} I\otimes A_0 & I\otimes A_1 \\ -A_1^T\otimes I & -A_0^T\otimes I \end{pmatrix}.$$

Boundary value conditions (6.20) take the form

$$M\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} + N\begin{pmatrix} y(h) \\ z(h) \end{pmatrix} = -\begin{pmatrix} 0 \\ w \end{pmatrix}, \tag{6.29}$$

where $w = vec(W)$ and

$$M = \begin{pmatrix} I \otimes I & 0_{n\times n} \otimes 0_{n\times n} \\ I \otimes A_0 - A_1^T \otimes D & I \otimes A_1 - A_0^T \otimes D \end{pmatrix},$$

$$N = \begin{pmatrix} 0_{n\times n} \otimes 0_{n\times n} & -I \otimes I \\ A_1^T \otimes I - D^T \otimes A_0 & A_0^T \otimes I - D^T \otimes A_1 \end{pmatrix}.$$

It follows from system (6.28) that

$$\begin{pmatrix} y(h) \\ z(h) \end{pmatrix} = \mathrm{e}^{Lh} \begin{pmatrix} y(0) \\ z(0) \end{pmatrix}.$$

Substituting the preceding equality into boundary value condition (2.29) we obtain an algebraic system for the initial vectors

$$\left[M + N\mathrm{e}^{Lh} \right] \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = - \begin{pmatrix} 0 \\ w \end{pmatrix}. \tag{6.30}$$

Under the Lyapunov condition the algebraic system admits a unique solution, which generates the corresponding solution of system (6.28),

$$\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = \mathrm{e}^{L\tau} \begin{pmatrix} y(0) \\ z(0) \end{pmatrix},$$

and a solution $(Y(\tau), Z(\tau))$ of boundary value problem (6.19)–(6.20). Finally, we arrive at the Lyapunov matrix associated with W:

$$U(\tau) = Y(\tau), \quad \tau \in [0, h].$$

We conclude this section with a criterion that system (6.1) satisfies the Lyapunov condition.

Theorem 6.8. *System (6.1) satisfies the Lyapunov condition if and only if the following condition holds:*

$$\det \left(M + N\mathrm{e}^{Lh} \right) \neq 0.$$

Proof. The proof is similar to that of Theorem 2.10. □

6.7 Spectral Properties

The spectrum of system (6.1) consists of all complex numbers s for which the characteristic matrix of the system,

$$G(s) = s\left(I - \mathrm{e}^{-sh} D \right) - A_0 - \mathrm{e}^{-sh} A_1,$$

is singular:

$$\Lambda = \{s_0 \mid \det G(s_0) = 0\}.$$

The spectrum of system (6.19) consists of all complex numbers s for which the system of algebraic matrix equations

$$\begin{cases} P(sI - A_0) - Q(sD + A_1) = 0_{n\times n} \\ \left(-sD^T + A_1^T\right)P + \left(sI + A_0^T\right)Q = 0_{n\times n} \end{cases} \tag{6.31}$$

admits a nontrivial solution (P,Q).

Lemma 6.10. *The spectrum of system (6.19) is symmetric with respect to the origin.*

Proof. Let s be an eigenvalue of system (6.19). Then there exists a nontrivial solution (P,Q) of (6.31). If we transpose these equalities, then it becomes evident that the pair $(P_1,Q_1) = \left(Q^T,P^T\right)$ satisfies the equations

$$\begin{cases} P_1\left((-s)I - A_0\right) - Q_1\left((-s)D + A_1\right) = 0_{n\times n}, \\ \left(-(-s)D^T + A_1^T\right)P_1 + \left((-s)I + A_0^T\right)Q_1 = 0_{n\times n}. \end{cases}$$

Since the new pair is nontrivial, $-s$ is an eigenvalue of system (6.19). □

Theorem 6.9. *Let system (6.1) have an eigenvalue s_0 such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of system (6.19).*

Proof. Observe first that there exist two nontrivial vectors γ and μ such that

$$\gamma^T G(s_0) = \gamma^T\left(s_0 I - A_0\right) - \mathrm{e}^{-s_0 h}\gamma^T\left(s_0 D + A_1\right) = 0$$

and

$$G^T(-s_0)\mu = \left(-s_0 I - A_0^T\right)\mu - \mathrm{e}^{s_0 h}\left(-s_0 D^T + A_1^T\right)\mu = 0.$$

Let us multiply the first equality by μ from the left-hand side,

$$\mu\gamma^T\left(s_0 I - A_0\right) - \mathrm{e}^{-s_0 h}\mu\gamma^T\left(s_0 D + A_1\right) = 0_{n\times n},$$

and the second one by $-\mathrm{e}^{-s_0 h}\gamma^T$ from the right-hand side,

$$\left(s_0 I + A_0^T\right)\mathrm{e}^{-s_0 h}\mu\gamma^T + \left(-s_0 D^T + A_1^T\right)\mu\gamma^T = 0_{n\times n}.$$

This means that the matrices $P = \mu\gamma^T$ and $Q = \mathrm{e}^{-s_0 h}\mu\gamma^T$ satisfy system (6.31) for $s = s_0$. As these matrices are nontrivial, we arrive at the final conclusion that s_0 belongs to the spectrum of system (6.19). The same is true for $-s_0$. □

6.8 A New Form for Lyapunov Functionals

Here we present functional (6.12) in a new form. The main feature of this new form is that it does not involve the derivative of the function $\varphi(\theta)$. In other words, we transform all terms of the functional that include the derivative in such a way that the new expressions for them do not include it. We assume in what follows that the initial functions are continuously differentiable, $\varphi \in C^1([-h,0],R^n)$.

Functional (6.12) can be written as

$$
\begin{aligned}
v_0(\varphi) = {} & \varphi^T(0)\left[U(0) - D^T U(h) - U(-h)D + D^T U(0) D\right]\varphi(0) \\
& + \underline{2\varphi^T(0)\int_{-h}^{0}\left[U(h+\theta) - U(\theta)D\right]^T D\varphi'(\theta)\mathrm{d}\theta} \\
& + 2\varphi^T(0)\int_{-h}^{0}\left[U(h+\theta) - U(\theta)D\right]^T A_1\varphi(\theta)\mathrm{d}\theta \\
& + \int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)A_1\varphi(\theta_2)]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
& + \underline{2\int_{-h}^{0}\left[D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1} \\
& + \underline{\int_{-h}^{0}\left[D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)D\varphi'(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1}.
\end{aligned}
$$

Here we underline the terms that depend on $\varphi'(\theta)$.

The first one is

$$
\begin{aligned}
J_1 = {} & 2\varphi^T(0)\int_{-h}^{0}\left[U(h+\theta) - U(\theta)D\right]^T D\varphi'(\theta)\mathrm{d}\theta \\
= {} & 2\varphi^T(0)\left[U(-h)D - D^T U(0)D\right]\varphi(0) \\
& - 2\varphi^T(0)\left[U(0)D - D^T U(h)D\right]\varphi(-h) \\
& - 2\varphi^T(0)\int_{-h}^{0}\left[U(h+\theta)A_0 + U(\theta)A_1\right]^T D\varphi(\theta)\mathrm{d}\theta.
\end{aligned}
$$

Now, we transform the second one,

$$J_2 = 2\int_{-h}^{0} \left[D\varphi'(\theta_1)\right]^T \left(\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$

$$= 2\varphi^T(0)\int_{-h}^{0} D^T U(-\theta)A_1\varphi(\theta)\mathrm{d}\theta - 2\varphi^T(-h)\int_{-h}^{0} D^T U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta$$

$$-2\int_{-h}^{0}\varphi^T(\theta_1)\left(\int_{-h}^{0} D^T U'(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1.$$

Finally, we address the term

$$J_3 = \int_{-h}^{0}\left[D\varphi'(\theta_1)\right]^T \left(\underbrace{\int_{-h}^{0} U(\theta_1-\theta_2)D\varphi'(\theta_2)\mathrm{d}\theta_2}_{R}\right)\mathrm{d}\theta_1.$$

First, we transform the internal integral

$$R = U(\theta_1)D\varphi(0) - U(\theta_1+h)D\varphi(-h) + \int_{-h}^{0} U'(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2,$$

hence the term

$$J_3 = \underbrace{\left(\int_{-h}^{0}\left[D\varphi'(\theta)\right]^T U(\theta)D\mathrm{d}\theta\right)\varphi(0)}_{J_{31}} - \underbrace{\left(\int_{-h}^{0}\left[D\varphi'(\theta)\right]^T U(\theta+h)D\mathrm{d}\theta\right)\varphi(-h)}_{J_{32}}$$

$$+\underbrace{\int_{-h}^{0}\left[D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0} U'(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1}_{J_{33}}.$$

Here

$$J_{31} = \left(\int_{-h}^{0}\left[D\varphi'(\theta)\right]^T U(\theta)D\mathrm{d}\theta\right)\varphi(0)$$

$$= \varphi^T(0)D^T U(0)D\varphi(0) - \varphi^T(-h)D^T U(-h)D\varphi(0)$$

$$
- \left(\int_{-h}^{0} \varphi^T(\theta) D^T U'(\theta) D \mathrm{d}\theta \right) \varphi(0),
$$

$$
\begin{aligned}
J_{32} = & - \left(\int_{-h}^{0} \left[D\varphi'(\theta) \right]^T U(\theta + h) D \mathrm{d}\theta \right) \varphi(-h) \\
= & - \varphi^T(0) D^T U(h) D\varphi(-h) + \varphi^T(-h) D^T U(0) D\varphi(-h) \\
& + \left(\int_{-h}^{0} \varphi^T(\theta) D^T U'(h + \theta) D \mathrm{d}\theta \right) \varphi(-h),
\end{aligned}
$$

and

$$
\begin{aligned}
J_{33} = & \int_{-h}^{0} \left[D\varphi'(\theta_1) \right]^T \left(\int_{-h}^{0} U'(\theta_1 - \theta_2) D\varphi(\theta_2) \right) \mathrm{d}\theta_1 \\
= & \varphi^T(0) \int_{-h}^{0} D^T U'(-\theta) D\varphi(\theta) \mathrm{d}\theta - \varphi^T(-h) \int_{-h}^{0} D^T U'(-h - \theta) D\varphi(\theta) \mathrm{d}\theta \\
& - \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} D^T U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1.
\end{aligned}
$$

Remark 6.7. In the computation of I_{33} one must remember that $U'(\tau)$ suffers discontinuity at $\tau = 0$; therefore

$$
\begin{aligned}
& - \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} D^T U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\
& = - \int_{-h}^{0} \varphi^T(\theta_1) D^T \left[\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right. \\
& \quad \left. + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 \\
& \quad - \int_{-h}^{0} \varphi^T(\theta) D^T \left[U'(+0) - U'(-0) \right] D\varphi(\theta) \mathrm{d}\theta.
\end{aligned}
$$

And we arrive at the following expression for the term J_3:

$$
\begin{aligned}
J_3 = {} & \varphi^T(0)D^TU(0)D\varphi(0) - 2\varphi^T(0)D^TU(h)D\varphi(-h) + \varphi^T(-h)D^TU(0)D\varphi(-h) \\
& +2\varphi^T(0)D^T\int_{-h}^{0}U'(-\theta)D\varphi(\theta)\mathrm{d}\theta - 2\varphi^T(-h)D^T\int_{-h}^{0}U'(-\theta-h)D\varphi(\theta)\mathrm{d}\theta \\
& -\int_{-h}^{0}\varphi^T(\theta_1)D^T\left(\int_{-h}^{\theta_1-0}U''(\theta_1-\theta_2)\varphi(\theta_2)\mathrm{d}\theta_2 + \int_{\theta_1+0}^{0}U''(\theta_1-\theta_2)\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
& -\int_{-h}^{0}\varphi^T(\theta)D^T\left[U'(+0)-U'(-0)\right]D\varphi(\theta)\mathrm{d}\theta.
\end{aligned}
$$

Substituting these expressions for J_1, J_2, J_3 in $v_0(\varphi)$ and collecting similar terms we arrive at the desired new form of the functional

$$
\begin{aligned}
v_0(\varphi) = {} & [\varphi(0)-D\varphi(-h)]^T U(0)[\varphi(0)-D\varphi(-h)] \\
& +2[\varphi(0)-D\varphi(-h)]^T\int_{-h}^{0}\left[U'(-h-\theta)D + U(-h-\theta)A_1\right]\varphi(\theta)\mathrm{d}\theta \\
& +\int_{-h}^{0}\varphi^T(\theta_1)\left(\int_{-h}^{0}A_1^TU(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
& -\int_{-h}^{0}\varphi^T(\theta_1)\left(\int_{-h}^{0}\left[D^TU'(\theta_1-\theta_2)A_1 - A_1^TU'(\theta_1-\theta_2)D\right]\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
& -\int_{-h}^{0}\varphi^T(\theta_1)D^T\left(\int_{-h}^{\theta_1-0}U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\right. \\
& \left. +\int_{\theta_1+0}^{0}U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
& -\int_{-h}^{0}\varphi^T(\theta)D^TPD\varphi(\theta)\mathrm{d}\theta. \qquad (6.32)
\end{aligned}
$$

Here P is the solution of the Schur matrix equation (6.10) (Remark 6.4).

Remark 6.8. It follows from Eq. (6.26) that

$$\begin{aligned}&A_1^T U(\tau)A_1 - D^T U'(\tau)A_1 + A_1^T U'(\tau)D - D^T U''(\tau)D\\&\quad = A_0^T U(\tau)A_0 + U'(\tau)A_0 - A_0^T U'(\tau) - U''(\tau), \quad \tau \in [0,h],\end{aligned}$$

so the functional can also be written as

$$\begin{aligned}v_0(\varphi) = {} & [\varphi(0) - D\varphi(-h)]^T U(0)[\varphi(0) - D\varphi(-h)]\\&+2[\varphi(0) - D\varphi(-h)]^T \int_{-h}^{0} \left[U'(-h-\theta)D + U(-h-\theta)A_1\right]\varphi(\theta)\mathrm{d}\theta\\&+\int_{-h}^{0} \varphi^T(\theta_1)\left(\int_{-h}^{0} \left[A_0^T U(\theta_1-\theta_2)A_0 + U'(\theta_1-\theta_2)A_0\right.\right.\\&\left.\left.-A_0^T U'(\theta_1-\theta_2)\right]\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1\\&-\int_{-h}^{0} \varphi^T(\theta_1)D^T\left(\int_{-h}^{\theta_1-0} U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\right.\\&\left.+\int_{\theta_1+0}^{0} U''(\theta_1-\theta_2)D\varphi(\theta_2)]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 - \int_{-h}^{0} \varphi^T(\theta)D^T PD\varphi(\theta)\mathrm{d}\theta.\end{aligned}$$

6.9 Complete Type Functionals

Given symmetric matrices W_j, $j = 0,1,2$, one can define the functional

$$\begin{aligned}w(\varphi) = {} & \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h)\\&+\int_{-h}^{0} \varphi^T(\theta)W_2\varphi(\theta)\mathrm{d}\theta, \quad \varphi \in PC^1([-h,0],R^n).\end{aligned}$$

Theorem 6.10. *Let system (6.1) satisfy the Lyapunov condition. Then the functional*

$$v(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta, \quad \varphi \in PC^1([-h,0],R^n), \tag{6.33}$$

where $v_0(\varphi)$ *is defined by (6.32) with the Lyapunov matrix* $U(\tau)$ *associated with* $W = W_0 + W_1 + hW_2$*, is such that*

$$\frac{\mathrm{d}}{\mathrm{dt}} v(x_t) = -w(x_t), \quad t \geq 0,$$

along the solutions of the system.

Proof. The proof is similar to that of Theorem 2.11. □

Remark 6.9. Functional (6.33) can be written as

$$\begin{aligned}
v(\varphi) = {} & [\varphi(0) - D\varphi(-h)]^T U(0) [\varphi(0) - D\varphi(-h)] \\
& + 2[\varphi(0) - D\varphi(-h)]^T \int_{-h}^{0} \left[U'(-h-\theta)D + U(-h-\theta)A_1\right] \varphi(\theta)\mathrm{d}\theta \\
& + \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} A_1^T U(\theta_1 - \theta_2) A_1 \varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\
& + \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} \left[A_1^T U'(\theta_1 - \theta_2) D - D^T U'(\theta_1 - \theta_2) A_1\right] \varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\
& - \int_{-h}^{0} \varphi^T(\theta_1) D^T \left(\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \right. \\
& \left. + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\
& + \int_{-h}^{0} \varphi^T(\theta) \left(W_1 + (h+\theta)W_2 - D^T P D\right) \varphi(\theta)\mathrm{d}\theta. \qquad (6.34)
\end{aligned}$$

Definition 6.7. We say that functional (6.33) [(6.34)] is of the complete type if the matrices W_j, $j = 0,1,2$, are positive definite.

6.10 Quadratic Bounds

Lemma 6.11. *Let system (6.1) be exponentially stable. If the matrices* W_j*,* $j = 0,1,2$*, are positive definite, then there exists* $\alpha_1 > 0$ *such that the complete type functional (6.34) satisfies the inequality*

$$\alpha_1 \|\varphi(0) - D\varphi(-h)\|^2 \leq v(\varphi), \quad \varphi \in PC^1([-h,0],R^n).$$

Proof. Consider the functional

$$\widetilde{v}(\varphi) = v(\varphi) - \alpha \left\| \varphi(0) - D\varphi(-h) \right\|^2 .$$

Its time derivative along the solutions of system (6.1) is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t), \quad t \geq 0,$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\alpha \left[x(t) - Dx(t-h)\right]^T \left[A_0 x(t) + A_1 x(t-h)\right] \\ &\geq \left[x^T(t), x^T(t-h)\right] L(\alpha) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}\end{aligned}$$

and the matrix

$$L(\alpha) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \alpha \begin{pmatrix} A_0 + A_0^T & A_1 - A_0^T D \\ A_1^T - D^T A_0 & -D^T A_1 - A_1^T D \end{pmatrix}.$$

It is evident that there exists $\alpha = \alpha_1 > 0$ such that the matrix $L(\alpha_1)$ is positive definite. For $\alpha = \alpha_1$ the inequality $\widetilde{w}(x_t) \geq 0$ holds, and as a consequence we conclude that

$$\widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0.$$

The preceding inequality implies that

$$\alpha_1 \left\| \varphi(0) - D\varphi(-h) \right\|^2 \leq v(\varphi).$$

□

Lemma 6.12. *Let system (6.1) satisfy the Lyapunov condition. Given the symmetric matrices W_0, W_1, and W_2, there exists $\alpha_2 > 0$ such that functional (6.34) satisfies the inequality*

$$v(\varphi) \leq \alpha_2 \left\| \varphi \right\|_h^2, \quad \varphi \in PC^1([-h,0],R^n).$$

Proof. Let us introduce the following quantities:

$$u_0 = \left\| U(0) \right\|, \quad u_1 = \sup_{\tau\in(0,h)} \left\| -D^T U'(\tau) + A_1^T U(\tau) \right\|,$$

and

$$u_2 = \sup_{\tau\in(0,h)} \left\| -D^T U''(\tau) D_1 + A_1^T U(\tau) A_1 - D^T U'(\tau) A_1 + A_1^T U'(\tau) D \right\|.$$

The first term in (6.34) admits the upper estimation

$$R_1 = [\varphi(0) - D\varphi(-h)]^T U(0) [\varphi(0) - D\varphi(-h)] \leq u_0 (1 + \|D\|)^2 \|\varphi\|_h^2.$$

The second term in (6.34) can be estimated as follows:

$$\begin{aligned} R_2 &= 2 [\varphi(0) - D\varphi(-h)]^T \int_{-h}^{0} \left[-D^T U'(h+\theta) + A_1^T U(h+\theta) \right]^T \varphi(\theta) \mathrm{d}\theta \\ &\leq 2hu_1 (1 + \|D\|) \|\varphi\|_h^2. \end{aligned}$$

The double integral admits the estimation

$$\begin{aligned} R_3 = \int_{-h}^{0} \varphi^T(\theta_1) \Bigg(\int_{-h}^{0} & \left[A_1^T U(\theta_1 - \theta_2) A_1 - D^T U'(\theta_1 - \theta_2) A_1 \right. \\ & \left. + A_1^T U'(\theta_1 - \theta_2) D \right] \varphi(\theta_2) \mathrm{d}\theta_2 \Bigg) \mathrm{d}\theta_1 \\ & - \int_{-h}^{0} \varphi^T(\theta_1) D^T \left[\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right. \\ & \left. + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 \\ \leq h^2 u_2 \|\varphi\|_h^2. & \end{aligned}$$

Finally, the term

$$\begin{aligned} R_4 &= \int_{-h}^{0} \varphi^T(\theta) \left[W_1 + (h+\theta) W_2 + D^T P D \right] \varphi(\theta) \mathrm{d}\theta \\ &\leq h \left(\|W_1\| + h \|W_2\| + \left\| D^T P D \right\| \right) \|\varphi\|_h^2. \end{aligned}$$

As a result, we arrive at the following quadratic upper bound for the functional

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2,$$

where

$$\alpha_2 = u_0 (1 + \|D\|)^2 + 2hu_1 (1 + \|D\|) + h^2 u_2 + h \|W_1\| + h \|W_2\| + h \left\| D^T P D \right\|.$$

□

We derive now slightly different upper and lower quadratic bounds for the functionals.

Lemma 6.13. *Let system (6.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, there exist $\beta_j > 0$, $j = 1,2$, such that the complete type functional (6.34) satisfies the inequality*

$$\beta_1 \|\varphi(0) - D\varphi(-h)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \le v(\varphi), \quad \varphi \in PC^1([-h,0],R^n).$$

Proof. Consider the functional

$$\widetilde{v}(\varphi) = v(\varphi) - \beta_1 \|\varphi(0) - D\varphi(-h)\|^2 - \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

Its time derivative along the solutions of system (6.1) is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t),$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\beta_1 [x(t) - Dx(t-h)]^T [A_0 x(t) + A_1 x(t-h)] \\ &\quad + \beta_2 \left[\|x(t)\|^2 - \|x(t-h)\|^2\right] \\ &\ge \left[x^T(t), x^T(t-h)\right] L(\beta_1,\beta_2) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}\end{aligned}$$

and the matrix

$$\begin{aligned}L(\beta_1,\beta_2) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \beta_1 \begin{pmatrix} A_0 + A_0^T & A_1 - A_0^T D \\ A_1^T - D^T A_0 & -D^T A_1 - A_1^T D \end{pmatrix} \\ + \beta_2 \begin{pmatrix} I & 0_{n\times n} \\ 0_{n\times n} & -I \end{pmatrix}.\end{aligned}$$

The matrix

$$L(0,0) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix}$$

is positive definite, so there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that $L(\beta_1,\beta_2) \ge 0$. For these values of β_1 and β_2, $\widetilde{w}(x_t) \ge 0$ and

$$\widetilde{v}(\varphi) = \int_0^{\infty} \widetilde{w}(x_t(\varphi))\mathrm{d}t \ge 0.$$

The last inequality justifies the statement of the lemma. □

Lemma 6.14. *Let system (6.1) satisfy the Lyapunov condition. Given the symmetric matrices W_0, W_1, and W_2, there exist $\delta_j > 0,\ j = 1,2$, such that functional (6.34) satisfies the inequality*

$$v(\varphi) \leq \delta_1 \|\varphi(0) - D\varphi(-h)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC^1([-h,0],R^n).$$

Proof. Using notations introduced in the proof of Lemma 6.12 we derive the following inequalities:

$$R_1 = [\varphi(0) - D\varphi(-h)]^T U(0) [\varphi(0) - D\varphi(-h)] \leq u_0 \|\varphi(0) - D\varphi(-h)\|^2,$$

$$R_2 = 2[\varphi(0) - D\varphi(-h)]^T \int_{-h}^{0} \left[-D_1^T U'(h+\theta) + A_1^T U(h+\theta)\right]^T \varphi(\theta)\mathrm{d}\theta$$

$$\leq u_1 \left[h \|\varphi(0) - D\varphi(-h)\|^2 + \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \right],$$

$$R_3 = \int_{-h}^{0} \varphi^T(\theta) \left[W_1 + (h+\theta)W_2 + D^T P D\right] \varphi(\theta)\mathrm{d}\theta$$

$$\leq \left(\|W_1\| + h\|W_2\| + \left\|D^T P D\right\|\right) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

$$R_4 = \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} \left[A_1^T U(\theta_1 - \theta_2) A_1 - D^T U'(\theta_1 - \theta_2) A_1 \right.\right.$$

$$\left.\left. + A_1^T U'(\theta_1 - \theta_2) D\right] \varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$- \int_{-h}^{0} \varphi^T(\theta_1) D^T \left[\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \right.$$

$$\left. + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

$$\leq h u_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

Hence, we arrive at the inequality

$$v(\varphi) \leq \delta_1 \|\varphi(0) - D\varphi(-h)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where

$$\delta_1 = u_0 + hu_1, \text{ and } \delta_2 = u_1 + \|W_1\| + h\|W_2\| + \left\|D^T PD\right\| + hu_2. \qquad \square$$

6.11 Applications

We present here some applications of the Lyapunov matrices and quadratic functionals studied in this chapter.

6.11.1 Exponential Estimates

In this section we apply the complete type functionals to derive exponential estimates for the solutions of system (6.1).

Lemma 6.15. *Let system (6.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, there exists $\sigma_1 > 0$ such that the complete type functional (6.34) satisfies the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) + 2\sigma_1 v(x_t) \leq 0, \quad t \geq 0,$$

along the solutions of the system.

Proof. On the one hand, we have for $\sigma_1 > 0$ the inequality

$$2\sigma_1 v(x_t) \leq 2\sigma_1 \delta_1 \|x(t) - Dx(t-h)\|^2 + 2\sigma_1 \delta_2 \int_{-h}^{0} \|x(t+\theta)\|^2 \mathrm{d}\theta, \quad t \geq 0,$$

where δ_1 and δ_2 are defined in Lemma 6.14.

On the other hand, we know that

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) = -x^T(t) W_0 x(t) - x^T(t-h) W_1 x(t-h) - \int_{-h}^{0} x^T(t+\theta) W_2 x(t+\theta) \mathrm{d}\theta.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t)+2\sigma_1 v(x_t) \leq -\left[x^T(t),x^T(t-h)\right]L(\sigma_1)\begin{bmatrix}x(t)\\x(t-h)\end{bmatrix} - \int_{-h}^{0} x^T(t+\theta)\left[W_2-2\sigma_1\delta_2 I\right]x(t+\theta)\mathrm{d}\theta, \quad t\geq 0.$$

Here the matrix

$$L(\sigma_1)=\begin{pmatrix}W_0 & 0_{n\times n}\\ 0_{n\times n} & W_1\end{pmatrix}-2\sigma_1\delta_1\begin{pmatrix}I & -D\\ -D^T & D^TD\end{pmatrix}.$$

Thus, if σ_1 satisfies the inequalities

$$L(\sigma_1)\geq 0, \text{ and } W_2\geq 2\sigma_1\delta_2 I,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t)+2\sigma_1 v(x_t)\leq 0, \quad t\geq 0.$$

□

Corollary 6.5. *Lemma 6.15 and Lemmas 6.11 and 6.12 imply the following exponential estimate for the solutions of system (6.1):*

$$\|x(t,\varphi)-Dx(t-h,\varphi)\|\leq \mu\,\|\varphi\|_h\,\mathrm{e}^{-\sigma_1 t}, \quad t\geq 0,$$

where

$$\mu=\sqrt{\frac{\alpha_2}{\alpha_1}}.$$

Lemma 6.16. *Consider the system*

$$x(t)-Dx(t-h)=f(t), \quad t\geq 0, \tag{6.35}$$

where

$$\|f(t)\|\leq \mu\,\|\varphi\|_h\,\mathrm{e}^{-\sigma_1 t}, \quad t\geq 0.$$

Let the matrix D be Schur stable; then there exist $\gamma>0$ and $\sigma>0$ such that the solution $x(t,\varphi)$, $\varphi\in PC^1([-h,0],R^n)$, of system (6.35) satisfies the inequality

$$\|x(t,\varphi)\|\leq\gamma\|\varphi\|_h\,\mathrm{e}^{-\sigma t}, \quad t\geq 0.$$

Proof. The matrix D is Schur stable, so there exist $d\geq 1$ and $\rho\in(0,1)$ such that $\left\|D^j\right\|\leq \mathrm{d}\rho^j$, $j\geq 0$. There exists $\sigma_2>0$ for which $\rho=\mathrm{e}^{-h\sigma_2}$.

For a given $t \geq 0$ we define the integer k such that $t \in [(k-1)h, kh)$. Iterating Eq. (6.35) $k-1$ times we arrive at the equality

$$x(t,\varphi) = D^k \varphi(\xi) + \sum_{j=0}^{k-1} D^j f(t-jh). \tag{6.36}$$

Now equality (6.36) implies that for $t \geq 0$

$$\|x(t,\varphi)\| \leq \mathrm{d}\left[\mathrm{e}^{-kh\sigma_2} + \mu \sum_{j=0}^{k-1} \mathrm{e}^{-jh\sigma_2}\mathrm{e}^{-(t-jh)\sigma_1}\right] \|\varphi\|_h.$$

If we select $\sigma_0 = \min\{\sigma_1, \sigma_2\}$, then $\mathrm{e}^{-(t-jh)\sigma_1} \leq \mathrm{e}^{-(t-jh)\sigma_0}$ and $\mathrm{e}^{-jh\sigma_2} \leq \mathrm{e}^{-jh\sigma_0}$, $j = 1, \dots, k-1$. Now we have

$$\|x(t,\varphi)\| \leq d\,[1+\mu k]\mathrm{e}^{-\sigma_0 t}\,\|\varphi\|_h.$$

Since $t \in [(k-1)\,h, kh)$, we have that $kh \leq t + h$ and

$$1 + \mu k \leq 1 + \mu + \frac{\mu}{h} t.$$

It is easy to verify that for any $\varepsilon > 0$

$$\max_{t \geq 0}\left\{t\mathrm{e}^{-\varepsilon t}\right\} = \frac{1}{\mathrm{e}\varepsilon},$$

so if we select

$$\sigma = \sigma_0 - \varepsilon, \text{ where } \varepsilon \in (0, \sigma_0), \text{ and } \gamma = \mathrm{d}\left[1 + \mu + \frac{\mu}{h\mathrm{e}\varepsilon}\right],$$

then we immediately arrive at the desired inequality. □

We are now able to state the main result of this section.

Theorem 6.11. *Let system (6.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, W_2, the solutions of the system satisfy the inequality*

$$\|x(t,\varphi)\| \leq \gamma \|\varphi\|_h \mathrm{e}^{-\sigma t}, \quad t \geq 0.$$

Here γ and σ are as computed in the proof of Lemma 6.16.

6.11.2 Quadratic Performance Index

We consider a control system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]=A_0x(t)+\widetilde{A}_1x(t-h)+Bu(t),\quad t\geq 0,$$
$$y(t)=Cx(t).$$

Given a control law

$$\widetilde{u}(t)=Mx(t-h),\quad t\geq 0, \tag{6.37}$$

a closed-loop system is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]=A_0x(t)+A_1x(t-h),\quad t\geq 0, \tag{6.38}$$

where $A_1=\widetilde{A}_1+BM$.

Assume that the closed-loop system is exponentially stable, and define the value of the quadratic performance index

$$J(\widetilde{u})=\int_0^{\infty}\left[y^T(t)Py(t)+u^T(t)Qu(t)\right]\mathrm{d}t. \tag{6.39}$$

Here P and Q are given symmetric matrices of the appropriate dimensions. The value of the index can now be written as

$$J(\widetilde{u})=\int_0^{\infty}\left[x^T(t,\varphi)W_0x(t,\varphi)+x^T(t-h,\varphi)W_1x(t-h,\varphi)\right]\mathrm{d}t,$$

where $\varphi\in PC^1([-h,0],R^n)$ is an initial function of the solution $x(t,\varphi)$ of closed-loop system (6.38) and the matrices $W_0=C^TPC$ and $W_1=M^TQM$.

Theorem 6.12. *The value of performance index (6.39) for the stabilizing control law (6.37) has the form*

$$J(\widetilde{u})=v_0(\varphi)+\int_{-h}^{0}\varphi^T(\theta)W_1\varphi(\theta)\mathrm{d}\theta,$$

where $v_0(\varphi)$ is the functional (6.12) computed with the Lyapunov matrix $U(\tau)$ associated with the matrix $W=W_0+W_1=C^TPC+M^TQM$.

6.11.3 Robustness Bounds

In this section we demonstrate how complete type functionals may be used for the robust stability analysis of a time-delay system. Consider the following perturbed system:

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)-Dy(t-h)]=(A_0+\Delta_0)\,y(t)+(A_1+\Delta_1)\,y(t-h),\quad t\geq 0,\tag{6.40}$$

where Δ_0 and Δ_1 are unknown but norm-bounded matrices

$$\left\|\Delta_j\right\|\leq r_j,\quad j=0,1.\tag{6.41}$$

Under the assumption that system (6.1) is exponentially stable, we derive bounds for r_0 and r_1 such that system (6.40) remains stable for all perturbation matrices satisfying (6.41).

We start with functional (6.34) computed for the nominal system (6.1). The time derivative of the functional along the solutions of system (6.40) is of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) = &-w(y_t)+2\,[y(t)-Dy(t-h)]^T\,U(0)\,[\Delta_0y(t)+\Delta_1y(t-h)]\\ &+2\,[\Delta_0y(t)+\Delta_1y(t-h)]^T\\ &\times\int_{-h}^{0}\left[U'(-h-\theta)D+U(-h-\theta)A_1\right]y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

If we introduce the values

$$u_0=\|U(0)\|,\quad u_1=\sup_{\tau\in(0,h)}\left\|-D^TU'(\tau)+A_1^TU(\tau)\right\|,$$

then the following inequalities hold:

$$\begin{aligned}J_1(t) &= 2\,[y(t)-Dy(t-h)]^T\,U(0)\,[\Delta_0y(t)+\Delta_1y(t-h)]\\ &\leq u_0\,[r_1+r_0(2+\|D\|)]\,\|y(t)\|^2+u_0\,[r_1+(r_0+2r_1)\,\|D\|]\,\|y(t-h)\|^2\end{aligned}$$

and

$$\begin{aligned}J_2(t) &= 2\,[\Delta_0y(t)+\Delta_1y(t-h)]^T\int_{-h}^{0}\left[-D^TU'(h+\theta)+A_1^TU(h+\theta)\right]^T y(t+\theta)\mathrm{d}\theta\\ &\leq hu_1r_0\,\|y(t)\|^2+hu_1r_1\,\|y(t-h)\|^2+u_1(r_0+r_1)\int_{-h}^{0}\|y(t+\theta)\|^2\,\mathrm{d}\theta.\end{aligned}$$

Therefore, we arrive at the statement.

Theorem 6.13. *Let the nominal system (6.1) be exponentially stable. Given the positive-definite matrices $W_j,\ j=0,1,2$, and the Lyapunov matrix $U(\tau)$ associated with $W=W_0+W_1+hW_2$, perturbed system (6.40) remains exponentially stable for all perturbations satisfying (6.41) if r_0 and r_1 satisfy the following inequalities:*

1. $\lambda_{\min}(W_0) > [r_1 + r_0(2+\|D\|)]\,u_0 + hr_0u_1,$
2. $\lambda_{\min}(W_1) > [r_1 + (r_0+2r_1)\,\|D\|]\,u_0 + hr_1u_1,$
3. $\lambda_{\min}(W_2) > (r_0+r_1)\,u_1.$

Corollary 6.6. *In Theorem 6.13 one may assume that the perturbation matrices Δ_0 and Δ_1 depend continuously on t and y_t.*

6.12 Notes and References

The classical volume [23] is a basic source of information about neutral type time-delay systems. It discusses, in a very general context, the principal properties of fundamental matrices and their application to the computation of the solutions of linear neutral type systems.

It seems that the first contribution dedicated to the computation of Lyapunov functionals with a given time derivative in the case of linear neutral type systems was written by Castelan and Infante [5]. In this contribution the authors first compute a quadratic functional for a difference approximation of system (6.1). Then the desired Lyapunov functional appears as a result of an appropriate limiting procedure. It is shown that the functional is determined by a special matrix valued function, the Lyapunov matrix in our terminology. The reader can find in this paper the three basic properties of a matrix valued function – the dynamic, the symmetry, and the algebraic – as well as the fact that the computation of the matrix is reduced to a special delay-free system of the form (6.19). The principal aim of this paper was to demonstrate that the presented functionals could be used in the computation of exponential estimates of the solutions of system (6.1). Unfortunately, this aim is not fulfilled since the attempt suffers on the same technical error as that in [28]; see Sect. 2.13 for details.

The observation that critical delay values can be computed on the basis of the spectrum of system (6.19) is made in [53].

Several aspects of the problem studied in this chapter are also discussed in the papers [32, 34, 35, 68].

It is worth mentioning that many of the results presented in this chapter can be extended to the case of systems with several delays multiple to a basic one at the expense of much more complicated formulas and expressions.

Chapter 7
Distributed Delay Case

This chapter is dedicated to the case of neutral type linear systems with distributed delay. The structure of quadratic functionals that have prescribed time derivatives along the solutions of such a system is defined, and the corresponding Lyapunov matrices are introduced. A system of matrix equations that defines Lyapunov matrices is given. It is proven that under some conditions this system admits a unique solution. A general class of system with distributed delay for which Lyapunov matrices are solutions of special standard boundary value problems for an auxiliary system of linear matrix ordinary differential equations is presented. Complete type functionals are defined. It is shown that these functionals can be presented in a special form that is more convenient for the computation of lower and upper bounds for the functionals.

7.1 Preliminaries

Let us consider the time-delay system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]=A_0x(t)+A_1x(t-h)+\int_{-h}^{0}G(\theta)x(t+\theta)\mathrm{d}\theta, \quad t\geq 0. \tag{7.1}$$

Here A_0, A_1, and D are given real $n\times n$ matrices, delay $h>0$, and $G(\theta)$ is a continuous matrix valued function defined for $\theta\in[-h,0]$.

Remark 7.1. It is worth noting that a system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)-\int_{-h}^{0}P(\theta)x(t+\theta)\mathrm{d}\theta\right]$$
$$=\widetilde{A}_0x(t)+\widetilde{A}_1x(t-h)+\int_{-h}^{0}\widetilde{G}(\theta)x(t+\theta)\mathrm{d}\theta$$

V.L. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*, Control Engineering, DOI 10.1007/978-0-8176-8367-2_7,

can be written as (7.1) with

$$A_0 = \widetilde{A}_0 + P(0), \quad A_1 = \widetilde{A}_1 - P(-h), \text{ and } G(\theta) = \widetilde{G}(\theta) - \frac{\mathrm{d}P(\theta)}{\mathrm{d}\theta}.$$

7.1.1 Fundamental Matrix

Let the $n \times n$ matrix $K(t)$ satisfy the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[K(t) - K(t-h)D\right] = K(t)A_0 + K(t-h)A_1 + \int_{-h}^{0} K(t+\theta)G(\theta)\mathrm{d}\theta, \quad t \geq 0,$$

with initial condition

$$K(t) = 0_{n\times n}, \quad t < 0, \quad K(0) = I,$$

and sewing condition

$$K(t) - K(t-h)D \text{ is continuous for } t \geq 0.$$

Matrix $K(t)$ is known as the *fundamental matrix* of system (7.1).

7.1.2 Cauchy Formula

Given an initial function $\varphi \in PC^1([-h,0],R^n)$, the corresponding solution, $x(t,\varphi)$, admits the following representation:

$$\begin{aligned} x(t,\varphi) = {} & \left[K(t) - K(t-h)D\right]\varphi(0) \\ & + \int_{-h}^{0} K(t-\theta-h)\left[A_1\varphi(\theta) + D\varphi'(\theta)\right]\mathrm{d}\theta \\ & + \int_{-h}^{0}\left(\int_{-h}^{\theta} K(t-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta, \quad t \geq 0. \end{aligned} \tag{7.2}$$

This expression is known as the Cauchy formula for system (7.1).

7.2 Lyapunov Functionals

Assume that system (7.1) is exponentially stable. Given a symmetric matrix W, there exists a quadratic functional $v_0(\varphi)$, defined on $PC^1([-h,0],R^n)$, such that along the solutions of system (7.1) the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

The functional may be presented as

$$v_0(\varphi) = \int_0^{\infty} x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t.$$

Substituting on the right-hand side of the last equality $x(t,\varphi)$ by formula (7.2) we arrive, after some direct calculations, at the following explicit expression for the quadratic functional:

$$\begin{aligned}
v_0(\varphi) &= \varphi^T(0)\left[U(0) - U(-h)D - D^T U(h) + D^T U(0)D\right]\varphi(0) \\
&+2\varphi^T(0)\int_{-h}^{0}\left[U(-h-\theta) - D^T U(-\theta)\right][A_1\varphi(\theta) + D\varphi'(\theta)]\mathrm{d}\theta \\
&+2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta}\left[U(-\theta+\xi) - D^T U(-\theta+\xi+h)\right]G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
&+\int_{-h}^{0}\left[A_1\varphi(\theta_1) + D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)\left[A_1\varphi(\theta_2) + D\varphi'(\theta_2)\right]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
&+2\int_{-h}^{0}\left[A_1\varphi(\theta_1) + D\varphi'(\theta_1)\right]^T \\
&\times\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2}U(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\
&+\int_{-h}^{0}\varphi^T(\theta_1)\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_1}G^T(\xi_1)\left(\int_{-h}^{\theta_2}U(\theta_1-\xi_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\right. \\
&\qquad\qquad\left.\times\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1. \qquad (7.3)
\end{aligned}$$

Here matrix

$$U(\tau) = \int_0^\infty K^T(t) W K(t+\tau) \mathrm{d}t, \quad \tau \in R, \tag{7.4}$$

is known as a Lyapunov matrix of system (7.1) associated with the matrix W.

7.3 Lyapunov Matrices

Formula (7.3) makes evident the importance of Lyapunov matrices in the construction of the Lyapunov functionals for system (7.1).

Theorem 7.1. *Given a symmetric matrix W, Lyapunov matrix (7.4) satisfies the following properties:*

- *Dynamic property*:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau) - U(\tau-h)D\right] = U(\tau)A_0 + U(\tau-h)A_1 + \int_{-h}^{0} U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau \geq 0; \tag{7.5}$$

- *Symmetry property*:

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0; \tag{7.6}$$

- *Algebraic property*:

$$\left[U'(+0) - U'(-0)\right] - D^T\left[U'(+0) - U'(-0)\right] D = -W. \tag{7.7}$$

Proof. The first two properties can be easily verified by direct calculation. We now address the third property. Differentiating the symmetry property (7.6) we obtain the equality

$$\frac{\mathrm{d}U(-\tau)}{\mathrm{d}\tau} = \left[\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right]^T, \quad \tau > 0.$$

In particular, when $\tau \to +0$, the equality takes the form $-U'(-0) = \left[U'(+0)\right]^T$.

Now, differentiating the product

$$J(t) = \left[K(t) - K(t-h)D\right]^T W \left[K(t) - K(t-h)D\right]$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}J(t) = [K(t) - K(t-h)D]^T W \left[K(t)A_0 + K(t-h)A_1 + \int_{-h}^{0} K(t+\theta)G(\theta)\mathrm{d}\theta \right]$$
$$+ \left[K(t)A_0 + K(t-h)A_1 + \int_{-h}^{0} K(t+\theta)G(\theta)\mathrm{d}\theta \right]^T W [K(t) - K(t-h)D].$$

Integrating the left-hand side of the last equality by t from 0 to ∞ we find that

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} [K(t) - K(t-h)D]^T W [K(t) - K(t-h)D] \mathrm{d}t = -W.$$

Compute now the integral of the first term on the right-hand side of the equality

$$\begin{aligned} J_1 &= \int_0^\infty [K(t) - K(t-h)D]^T W \\ &\quad \times \left[K(t)A_0 + K(t-h)A_1 + \int_{-h}^{0} K(t+\theta)G(\theta)\mathrm{d}\theta \right] \mathrm{d}t \\ &= \left[U(0)A_0 + U(-h)A_1 + \int_{-h}^{0} U(\theta)G(\theta)\mathrm{d}\theta \right] \\ &\quad - D^T \left[U(h)A_0 + U(0)A_1 + \int_{-h}^{0} U(h+\theta)G(\theta)\mathrm{d}\theta \right] \\ &= \left[U'(+0) - U'(-h+0)D \right] - D^T \left[U'(h-0) - U'(-0)D \right]. \end{aligned}$$

The value of the integral of the second term on the right-hand-side of the preceding equality is as follows:

$$\begin{aligned} J_2 &= \int_0^\infty \left[K(t)A_0 + K(t-h)A_1 + \int_{-h}^{0} K(t+\theta)G(\theta)\mathrm{d}\theta \right]^T \\ &\quad \times W [K(t) - K(t-h)D] \mathrm{d}t \\ &= \left[U'(+0) - U'(-h+0)D \right]^T - \left[U'(h-0) - U'(-0)D \right]^T D \\ &= -U'(-0) + D^T U'(h-0) + U'(-h+0)D - D^T U'(+0)D. \end{aligned}$$

And we arrive at equality (7.7). □

Remark 7.2. In explicit form property (7.7) has the form

$$
\begin{aligned}
-W &= U(0)A_0 + U(-h)A_1 + \int_{-h}^{0} U(\theta)G(\theta)\mathrm{d}\theta \\
&\quad -D^T\left[U(h)A_0 + U(0)A_1 + \int_{-h}^{0} U(h+\theta)G(\theta)\mathrm{d}\theta\right] \\
&\quad +A_0^T U(0) + A_1^T U(h) + \int_{-h}^{0} G^T(\theta)U(-\theta)\mathrm{d}\theta \\
&\quad -\left[A_0^T U(-h) + A_1^T U(0) + \int_{-h}^{0} G^T(\theta)U(-h-\theta)\mathrm{d}\theta\right] D. \qquad (7.8)
\end{aligned}
$$

7.4 Lyapunov Matrices: New Definition

In this section we continue our study of the Lyapunov matrices. We start with a new definition of the matrices that neither assumes the exponential stability of system (7.1) nor demands knowledge of the fundamental matrix of the system.

Definition 7.1. Given a symmetric matrix W, we say that the $n \times n$ matrix valued function $U(\tau)$ is called a Lyapunov matrix of system (7.1) associated with W if it satisfies properties (7.5)–(7.7).

Then we check that matrices satisfying Definition 7.1 can be used in formula (7.3).

Theorem 7.2. *Let matrix $U(\tau)$ satisfy Definition 7.1. Then functional (7.3) with the matrix is such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0,$$

along the solutions of system (7.1).

Proof. We present functional (7.3) in a form more suitable for the computation of the time derivative along the solutions of system (7.1):

$$
\begin{aligned}
v_0(x_t) &= x^T(t)\left[U(0) - U(-h)D - D^T U(h) + D^T U(0) D\right] x(t) \\
&\quad +2x^T(t)\int_{t-h}^{t} \left[U(\xi - t + h) - U(\xi - t)D\right]^T \left[A_1 x(\xi) + Dx'(\xi)\right]\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&+2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left[U(\xi-t-\theta)-U(\xi-t-h)D\right]^T G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi\\
&+\int_{t-h}^{t}\left[A_1x(\xi_1)+Dx'(\xi_1)\right]^T\\
&\times\left(\int_{t-h}^{t}U(\xi_1-\xi_2)\left[A_1x(\xi_2)+Dx'(\xi_2)\right]\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\\
&+2\int_{t-h}^{t}\left[A_1x(\xi_1)+Dx'(\xi_1)\right]^T\\
&\times\left[\int_{t-h}^{t}\left(\int_{-h}^{\xi_2-t}U(\xi_1+h-\xi_2+\theta)G(\theta)\mathrm{d}\theta\right)x(\xi_2)\mathrm{d}\xi_2\right]\mathrm{d}\xi_1\\
&+\int_{t-h}^{t}x^T(\xi_1)\left\{\int_{t-h}^{t}\left[\int_{-h}^{\xi_1-t}\left(\int_{-h}^{\xi_2-t}G^T(\theta_1)U(\xi_1-\xi_2-\theta_1+\theta_2)G(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1\right]\right.\\
&\left.\times x(\xi_2)\mathrm{d}\xi_2\right\}\mathrm{d}\xi_1.
\end{aligned}
$$

The time derivative of the first term,

$$
R_1(t)=x^T(t)\left[U(0)-U(-h)D-D^TU(h)+D^TU(0)D\right]x(t),
$$

is equal to

$$
\frac{\mathrm{d}R_1(t)}{\mathrm{d}t}=2x^T(t)\left[U(0)-U(-h)D-D^TU(h)+D^TU(0)D\right]x'(t).
$$

The time derivative of the second term,

$$
R_2(t)=2x^T(t)\int_{t-h}^{t}\left[U(\xi-t+h)-U(\xi-t)D\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi,
$$

is computed as follows:

$$
\begin{aligned}
\frac{\mathrm{d}R_2(t)}{\mathrm{d}t} &= \underline{2\left[x'(t)\right]^T \int_{t-h}^{t} \left[U(\xi-t+h)-U(\xi-t)D\right]^T \left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi} \\
&+ 2x^T(t)\left[U(-h)-D^TU(0)\right]\left[A_1x(t)+Dx'(t)\right] \\
&- 2x^T(t)\left[U(0)-D^TU(h)\right]\left[A_1x(t-h)+Dx'(t-h)\right] \\
&+ \underline{2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}\left[U(\xi-t+h)-U(\xi-t)D\right]^T\right)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi}.
\end{aligned}
$$

We now address the next term,

$$
R_3(t) = 2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left[U(\xi-t-\theta)-U(\xi-t-\theta-h)D\right]^T G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi.
$$

Its time derivative is

$$
\begin{aligned}
\frac{\mathrm{d}R_3}{\mathrm{d}t} &= \underline{\underline{2\left[x'(t)\right]^T\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left[U(\xi-t-\theta)-U(\xi-t-\theta-h)D\right]^T G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi}} \\
&+ 2x^T(t)\left(\int_{-h}^{0}\left[U(\theta)-D^TU(\theta+h)\right]G(\theta)\mathrm{d}\theta\right)x(t) \\
&- 2x^T(t)\left[U(0)-D^TU(h)\right]\int_{t-h}^{t}G(\xi-t)x(\xi)\mathrm{d}\xi \\
&+ \underline{\underline{2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left(\frac{\partial}{\partial t}\left[U(\xi-t-\theta)-U(\xi-t-\theta-h)D\right]\right)^T G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi}}.
\end{aligned}
$$

The time derivative of the fourth term,

$$
R_4(t) = \int_{t-h}^{t}\left[A_1x(\xi_1)+Dx'(\xi_1)\right]^T\left(\int_{t-h}^{t}U(\xi_1-\xi_2)\left[A_1x(\xi_2)+Dx'(\xi_2)\right]\mathrm{d}\xi_2\right)\mathrm{d}\xi_1,
$$

is equal to

$$\begin{aligned}\frac{\mathrm{d}R_4(t)}{\mathrm{d}t} &= \underline{2\left[A_1x(t)+Dx'(t)\right]^T \int_{t-h}^{t} U(t-\xi)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi} \\ &\quad \underline{-2\left[A_1x(t-h)+Dx'(t-h)\right]^T \int_{t-h}^{t} U(t-\xi-h)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi}.\end{aligned}$$

Now we differentiate the term

$$\begin{aligned}R_5(t) = 2\int_{t-h}^{t} &\left[A_1x(\xi_1)+Dx'(\xi_1)\right]^T \\ &\left[\int_{t-h}^{t}\left(\int_{-h}^{\xi_2-t} U(\xi_1+h-\xi_2+\theta)G(\theta)\mathrm{d}\theta\right)x(\xi_2)\mathrm{d}\xi_2\right]\mathrm{d}\xi_1.\end{aligned}$$

Here

$$\begin{aligned}\frac{\mathrm{d}R_5(t)}{\mathrm{d}t} &= \underline{\underline{2\left[A_1x(t)+Dx'(t)\right]^T \int_{t-h}^{t}\left(\int_{-h}^{\xi-t} U(t-\xi+h+\theta)G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi}} \\ &\quad \underline{\underline{-2\left[A_1x(t-h)+Dx'(t-h)\right]^T \int_{t-h}^{t}\left(\int_{-h}^{\xi-t} U(t-\xi+\theta)G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi}} \\ &\quad \underline{-2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{0} G^T(\theta)U(t-\xi-h-\theta)\mathrm{d}\theta\right)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi} \\ &\quad \underline{-2\left[\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T \int_{t-h}^{t} U(t-\xi-h)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi}.\end{aligned}$$

Finally, we address the term

$$\begin{aligned}R_6(t) = \int_{t-h}^{t} x^T(\xi_1)\Bigg\{\int_{t-h}^{t}\Bigg[\int_{-h}^{\xi_1-t}&\left(\int_{-h}^{\xi_2-t} G^T(\theta_1)U(\xi_1-\xi_2-\theta_1+\theta_2)G(\theta_2)\mathrm{d}\theta_2\right) \\ &\times \mathrm{d}\theta_1\Bigg]x(\xi_2)\mathrm{d}\xi_2\Bigg\}\mathrm{d}\xi_1.\end{aligned}$$

The time derivative of this term is as follows:

$$\frac{\mathrm{d}R_6(t)}{\mathrm{d}t} = \underline{\underline{2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{0}\left(\int_{-h}^{\xi-t} G^T(\theta_1)U(t-\xi-\theta_1+\theta_2)G(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1\right]x(\xi)\mathrm{d}\xi}}$$
$$\underline{\underline{-2\left[\int_{-h}^{0} G(\theta_1)x(t+\theta_1)\mathrm{d}\theta_1\right]^T\int_{t-h}^{t}\left(\int_{-h}^{\xi-t} U(t-\xi+\theta_2)G(\theta_2)\mathrm{d}\theta_2\right)x(\xi)\mathrm{d}\xi}}.$$

Now we collect in the computed time derivatives the terms underlined with one solid line. The sum of these terms is

$$\begin{aligned}
S_1(t) = {} & 2\left[x'(t)\right]^T\int_{t-h}^{t}\left[U(\xi-t+h)-U(\xi-t)D\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi \\
& + 2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}\left[U(\xi-t+h)-U(\xi-t)D\right]^T\right)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi \\
& + 2\left[A_1x(t)+Dx'(t)\right]^T\int_{t-h}^{t}\left[U(\xi-t)\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi \\
& - 2\left[A_1x(t-h)+Dx'(t-h)\right]^T\int_{t-h}^{t}\left[U(\xi-t+h)\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi \\
& - 2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{0} G^T(\theta)U(t-\xi-h-\theta)\mathrm{d}\theta\right)\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi \\
& - 2\left[\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T\int_{t-h}^{t}\left[U(\xi-t+h)\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi.
\end{aligned}$$

After simple rearrangement we obtain

$$\begin{aligned}
S_1(t) = {} & 2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]-A_1x(t-h)-\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T \\
& \times\int_{t-h}^{t}\left[U(\xi-t+h)\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

$$+\,2\left[-Dx'(t)+A_1x(t)+Dx'(t)\right]^T\int_{t-h}^{t}\left[U(\xi-t)\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi$$

$$+\,2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}\left[U(\xi-t+h)-U(\xi-t)D\right]+\int_{-h}^{0}U(\xi-t+h+\theta)G^T(\theta)\mathrm{d}\theta\right)^T$$
$$\times[A_1x(\xi)+Dx'(\xi)]\mathrm{d}\xi.$$

Because $x(t)$ is a solution of system (7.1), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]-A_1x(t-h)-\int_{-h}^{0}G(\theta)x(t+\theta)\mathrm{d}\theta=A_0x(t),\quad t\geq 0,$$

and we obtain

$$S_1(t)=2x^T(t)\int_{t-h}^{t}\left[U(\xi-t+h)A_0\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi$$
$$+\,2x^T(t)\int_{t-h}^{t}\left[U(\xi-t)A_1\right]^T\left[A_1x(\xi)+Dx'(\xi)\right]\mathrm{d}\xi$$
$$+\,2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}\left[U(\xi-t+h)-U(\xi-t)D\right]+\int_{-h}^{0}U(\xi-t+h+\theta)G^T(\theta)\mathrm{d}\theta\right)^T$$
$$\times[A_1x(\xi)+Dx'(\xi)]\mathrm{d}\xi.$$

Since $\xi-t+h\geq 0$ for $\xi\in[t-h,t]$, we have

$$\frac{\partial}{\partial t}\left[U(\xi-t+h)-U(\xi-t)D\right]=-U(\xi-t+h)A_0-U(\xi-t)A_1$$
$$-\int_{-h}^{0}U(\xi-t+h+\theta)G^T(\theta)\mathrm{d}\theta,$$

and we conclude that $S_1(t)=0$.

Now we collect in the time derivatives the double underlined terms. The sum of these terms is

$$S_2(t) = 2\left[x'(t)\right]^T \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} \left[U(\xi-t-\theta) - U(\xi-t-\theta-h)D\right]^T G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$+ 2x^T(t) \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} \left(\frac{\partial}{\partial t}\left[U(\xi-t-\theta)-U(\xi-t-\theta-h)D\right] \right)^T G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$+ 2\left[A_1x(t) + Dx'(t)\right]^T \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} U(t-\xi+h+\theta)G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$- 2\left[A_1x(t-h) + Dx'(t-h)\right]^T \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} U(t-\xi+\theta)G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$+ 2x^T(t) \int_{t-h}^{t} \left[\int_{-h}^{0} \left(\int_{-h}^{\xi-t} G^T(\theta_1)U(t-\xi-\theta_1+\theta_2)G(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \right] x(\xi)\mathrm{d}\xi$$

$$- 2\left[\int_{-h}^{0} G(\theta_1)x(t+\theta_1)\mathrm{d}\theta_1 \right]^T \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} U(t-\xi+\theta_2)G(\theta_2)\mathrm{d}\theta_2 \right) x(\xi)\mathrm{d}\xi.$$

Rearranging the terms we obtain

$$S_2(t) = 2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] - A_1x(t-h) - \int_{-h}^{0} G(\theta_1)x(t+\theta_1)\mathrm{d}\theta_1 \right]^T$$

$$\times \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} U(t-\xi+\theta)G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$+ 2\left[-Dx'(t) + A_1x(t) + Dx'(t)\right]^T \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} U(t-\xi+h+\theta)G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$+ 2x^T(t) \int_{t-h}^{t} \left(\int_{-h}^{\xi-t} \left(\frac{\partial}{\partial t}\left[U(\xi-t-\theta) - U(\xi-t-\theta-h)D\right] \right)^T G(\theta)\mathrm{d}\theta \right) x(\xi)\mathrm{d}\xi$$

$$+ 2x^T(t) \int_{t-h}^{t} \left[\int_{-h}^{0} \left(\int_{-h}^{\xi-t} G^T(\theta_1)U(t-\xi-\theta_1+\theta_2)G(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \right] x(\xi)\mathrm{d}\xi.$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]-A_1x(t-h)-\int_{-h}^{0}G(\theta_1)x(t+\theta_1)\mathrm{d}\theta_1=A_0x(t),\quad t\geq 0,$$

we have that

$$\begin{aligned}S_2(t) &= 2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left[U(\xi-t-\theta)A_0\right]G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi\\ &+ 2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left[U(\xi-t-h-\theta)A_1\right]G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi\\ &+ 2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{\xi-t}\left(\frac{\partial}{\partial t}\left[U(\xi-t-\theta)-U(\xi-t-\theta-h)D\right]\right)^T G(\theta)\mathrm{d}\theta\right)x(\xi)\mathrm{d}\xi\\ &+ 2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{\xi-t}\left(\int_{-h}^{0}U(\xi-t-\theta+\eta)G(\eta)\mathrm{d}\eta\right)^T G(\theta)\mathrm{d}\theta\right]x(\xi)\mathrm{d}\xi.\end{aligned}$$

Since $\xi-t-\theta\geq 0$ for $\xi\in[t-h,t]$ and $\theta\in[-h,\xi-t]$, property (7.5) implies the equality

$$\begin{aligned}\frac{\partial}{\partial t}\left[U(\xi-t-\theta)-U(\xi-t-\theta-h)D\right] &= -U(\xi-t-\theta)A_0-U(\xi-t-\theta-h)A_1\\ &\quad -\int_{-h}^{0}U(\xi-t-\theta+\theta_1)G(\theta_1)\mathrm{d}\theta_1,\end{aligned}$$

and we arrive at the conclusion that $S_2(t)=0$.

We now collect the nonunderlined terms:

$$\begin{aligned}S_3(t) &= 2x^T(t)\left[U(0)-U(-h)D-D^TU(h)+D^TU(0)D\right]x'(t)\\ &+ 2x^T(t)\left[U(-h)-D^TU(0)\right]\left[A_1x(t)+Dx'(t)\right]\\ &- 2x^T(t)\left[U(0)-D^TU(h)\right]\left[A_1x(t-h)+Dx'(t-h)\right]\\ &+ 2x^T(t)\left(\int_{-h}^{0}\left[U(\theta)-D^TU(\theta+h)\right]G(\theta)\mathrm{d}\theta\right)x(t)\\ &- 2x^T(t)\left[U(0)-D^TU(h)\right]\int_{t-h}^{t}G(\xi-t)x(\xi)\mathrm{d}\xi.\end{aligned}$$

Rearranging them as

$$\begin{aligned}S_3(t) &= 2x^T(t)\left[U(0)-D^TU(h)\right]\\&\quad\times\left[\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-Dx(t-h)\right]-A_1x(t-h)-\int_{-h}^{0}G(\theta)x(t+\theta)\mathrm{d}\theta\right]\\&\quad+2x^T(t)\left[U(-h)-D^TU(0)\right]\left[-Dx'(t)+A_1x(t)+Dx'(t)\right]\\&\quad+2x^T(t)\left(\int_{-h}^{0}\left[U(\theta)-D^TU(\theta+h)\right]G(\theta)\mathrm{d}\theta\right)x(t),\end{aligned}$$

we obtain the equality

$$\begin{aligned}S_3(t) &= 2x^T(t)\left[U(0)A_0+U(-h)A_1+\int_{-h}^{0}U(\theta)G(\theta)\mathrm{d}\theta\right]x(t)\\&\quad-2x^T(t)D^T\left[U(h)A_0+U(0)A_1+\int_{-h}^{0}U(\theta+h)G(\theta)\mathrm{d}\theta\right]x(t).\end{aligned}$$

Since

$$\begin{aligned}U(0)A_0+U(-h)A_1+\int_{-h}^{0}U(\theta)G(\theta)\mathrm{d}\theta &= \lim_{\tau\to+0}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau)-U(\tau-h)D\right]\\&=U'(+0)-U'(-h+0)D\end{aligned}$$

and

$$\begin{aligned}U(h)A_0+U(0)A_1+\int_{-h}^{0}U(\theta+h)G(\theta)\mathrm{d}\theta &= \lim_{\tau\to h-0}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau)-U(\tau-h)D\right]\\&=U'(h-0)-U'(-0)D,\end{aligned}$$

we have that

$$\begin{aligned}S_3(t) &= 2x^T(t)\left[U'(+0)-U'(-h+0)D\right]x(t)\\&\quad-2x^T(t)D^T\left[U'(h-0)-U'(-0)D\right]x(t).\end{aligned}$$

Property (7.6) implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(-\tau)=\left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\right]^T,$$

so

$$\left[U'(+0)-U'(-h+0)D\right]^T=-U'(-0)+D^TU'(h-0)$$

and

$$\left(D^T\left[U'(h-0)-U'(-0)D\right]\right)^T=-U'(-h+0)D+D^TU'(+0)D.$$

Simple symmetrization of the preceding expression for $S_3(t)$ shows that

$$\begin{aligned}S_3(t)&=x^T(t)\left(\left[U'(+0)-U'(-0)\right]-D^T\left[U'(+0)-U'(-0)\right]D\right)x(t)\\&=-x^T(t)Wx(t).\end{aligned}$$

The last equality follows directly from property (7.7).

The preceding computations demonstrate that

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t)=-x^T(t)Wx(t),\quad t\geq 0.$$ □

Finally, we prove that Definition 7.1 does not contradict the initial definition of Lyapunov matrices.

Lemma 7.1. *Let system (7.1) be exponentially stable. Then matrix (7.4) is a unique solution of delay matrix Eq. (7.5), which satisfies properties (7.6) and (7.7).*

Proof. Recall that matrix (7.4) satisfies properties (7.5)–(7.7); see Theorem 7.1.

Let there exist two matrices $U_1(\tau)$ and $U_2(\tau)$ that satisfy (7.5)–(7.7). We define two functionals (7.3), $v_0^{(j)}(\varphi)$, $j=1,2$, one with $U(\tau)=U_1(\tau)$, the other with $U(\tau)=U_2(\tau)$. By Theorem 7.2,

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0^{(j)}(x_t)=-x^T(t)Wx(t),\quad t\geq 0,\quad j=1,2.$$

Thus, the difference, $\Delta v_0(x_t)=v_0^{(2)}(x_t)-v_0^{(1)}(x_t)$, is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v_0(x_t)=0,\quad t\geq 0.$$

Integrating the last equality by t from 0 to $T\geq 0$ we obtain

$$\Delta v_0(x_T(\varphi))=\Delta v_0(\varphi),\quad T\geq 0.$$

System (7.1) is exponentially stable, so $x_T(\varphi) \to 0_h$ as $T \to \infty$, and we arrive at the conclusion that for any initial function $\varphi \in PC^1([-h,0],R^n)$ the following equality holds:

$$\Delta v_0(\varphi) = 0.$$

In explicit form this equality is written as

$$\begin{aligned}
0 = & \underbrace{\varphi^T(0)\left[\Delta U(0) - \Delta U(-h)D - D^T\Delta U(h) + D^T\Delta U(0)D\right]\varphi(0)}_{\Delta R_1} \\
& \underbrace{+ 2\varphi^T(0)\int_{-h}^{0}\left[\Delta U(-h-\theta) - D^T\Delta U(-\theta)\right][A_1\varphi(\theta) + D\varphi'(\theta)]\mathrm{d}\theta}_{\Delta R_2} \\
& \underbrace{+ 2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta}\left[\Delta U(-\theta+\xi) - D^T\Delta U(-\theta+\xi+h)\right]G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta}_{\Delta R_3} \\
& \underbrace{+ \int_{-h}^{0}\left[A_1\varphi(\theta_1) + D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0}\Delta U(\theta_1-\theta_2)\left[A_1\varphi(\theta_2) + D\varphi'(\theta_2)\right]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1}_{\Delta R_4} \\
& \underbrace{+ 2\int_{-h}^{0}\left[A_1\varphi(\theta_1) + D\varphi'(\theta_1)\right]^T\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2}\Delta U(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1}_{\Delta R_5} \\
& \underbrace{+ \int_{-h}^{0}\varphi^T(\theta_1)\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_1}\left(\int_{-h}^{\theta_2}G^T(\xi_1)\Delta U(\theta_1-\xi_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1}_{\Delta R_6}.
\end{aligned} \tag{7.9}$$

Here the matrix $\Delta U(\tau) = U_2(\tau) - U_1(\tau)$.

Let γ and μ be two constant vectors, $\tau \in (0,h]$, and $\varepsilon > 0$ is such that $-\tau + \varepsilon < 0$. Then we define the initial function

$$\varphi(\theta) = \begin{cases} \gamma, & \text{for } \theta = 0, \\ \mu, & \text{for } \theta \in [-\tau, -\tau+\varepsilon], \\ 0, & \text{at all other points of } [-h,0]. \end{cases}$$

Substitute this function into (7.9). The first term

$$\Delta R_1 = \gamma^T\left[\Delta U(0) - \Delta U(-h)D - D^T\Delta U(h) + D^T\Delta U(0)D\right]\gamma.$$

The second term

$$\begin{aligned}\Delta R_2 = {} & 2\gamma^T \left[\Delta U(-h) - D^T \Delta U(0)\right] D\gamma \\ & + 2\varepsilon\gamma^T \left[\Delta U(\tau-h)A_1 - D^T \Delta U(\tau)A_1 \right. \\ & \left. + \Delta U'(\tau-h)D - D^T \Delta U'(\tau)D\right] \mu + o(\varepsilon).\end{aligned}$$

Here $o(\varepsilon)$ stands for a quantity that satisfies the condition

$$\lim_{\varepsilon\to+0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Now the third term

$$\Delta R_3 = 2\varepsilon\gamma^T \left(\int_{-h}^{-\tau} \left[\Delta U(\tau+\xi) - D^T \Delta U(\tau+h+\xi)\right] G(\xi)\mathrm{d}\xi \right) \mu + o(\varepsilon).$$

The term

$$\begin{aligned}\Delta R_4 = {} & \gamma^T D^T \Delta U(0) D\gamma + 2\varepsilon\gamma^T D^T \left[\Delta U(\tau)A_1 + \Delta U'(\tau)D\right] \mu \\ & - \varepsilon\mu^T D^T \left[\Delta U'(+0) - \Delta U'(-0)\right] D\mu + o(\varepsilon).\end{aligned}$$

The fifth term

$$\Delta R_5 = 2\varepsilon\gamma^T D^T \left(\int_{-h}^{-\tau} \Delta U(\tau+h+\xi) G(\xi)\mathrm{d}\xi \right) \mu + o(\varepsilon).$$

Finally,

$$\Delta R_6 = o(\varepsilon).$$

For this initial function equality (7.9) takes the form

$$\begin{aligned}0 = {} & \gamma^T \Delta U(0)\gamma + 2\varepsilon\gamma^T \left(\Delta U(\tau-h)A_1 + \Delta U'(\tau-h)D + \int_{-h}^{-\tau} \Delta U(\tau+\xi) G(\xi)\mathrm{d}\xi \right) \mu \\ & - \varepsilon\mu^T D^T \left[\Delta U'(+0) - \Delta U'(-0)\right] D\mu + o(\varepsilon).\end{aligned}$$

In the preceding equality let $\mu = 0$. Then $0 = \gamma^T \Delta U(0)\gamma$, and since γ is an arbitrary vector and the matrix $\Delta U(0)$ is symmetric, we conclude that

$$\Delta U(0) = 0_{n\times n}. \tag{7.10}$$

For similar reasons

$$D^T \left[\Delta U'(+0) - \Delta U'(-0)\right] D = 0_{n\times n},$$

which in light of (7.7) means that

$$\Delta U'(+0) - \Delta U'(-0) = 0_{n\times n}.$$

Finally, because the vectors γ and μ are arbitrary, we conclude that

$$\Delta U(\tau - h)A_1 + \Delta U'(\tau - h)D + \int_{-h}^{-\tau} \Delta U(\tau + \xi)G(\xi)\mathrm{d}\xi = 0_{n\times n}, \quad \tau \in (0,h]. \tag{7.11}$$

By definition, the matrix $\Delta U(\tau)$ satisfies the equation

$$\begin{aligned}\Delta U'(\tau) - \Delta U'(\tau - h)D = \Delta U(\tau)A_0 + \Delta U(\tau - h)A_1 \\ + \int_{-h}^{0} \Delta U(\tau + \xi)G(\xi)\mathrm{d}\xi, \quad \tau \geq 0.\end{aligned}$$

Applying conditions (7.10) and (7.11), we arrive at the conclusion that the matrix $\Delta U(\tau)$ is a solution of the matrix equation

$$\begin{aligned}\Delta U'(\tau) &= \Delta U(\tau)A_0 + \int_{0}^{-\tau} \Delta U(\tau + \xi)G(\xi)\mathrm{d}\xi \\ &= \Delta U(\tau)A_0 + \int_{0}^{\tau} \Delta U(\theta)G(\theta - \tau)\mathrm{d}\theta, \quad \tau \geq 0,\end{aligned}$$

with the trivial initial condition $\Delta U(0) = 0_{n\times n}$. By Lemma 4.1, the solution is trivial, that is,

$$\Delta U(\tau) = U^{(2)}(\tau) - U^{(1)}(\tau) = 0_{n\times n}, \quad \tau \in [0,h]. \qquad \square$$

7.5 Existence Issue

The characteristic function of system (7.1) is of the form

$$f(s) = \det\left(sI - s\mathrm{e}^{-sh}D - A_0 - \mathrm{e}^{-sh}A_1 - \int_{-h}^{0} \mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right). \tag{7.12}$$

Here the matrix

$$H(s) = \left(sI - s\mathrm{e}^{-sh}D - A_0 - \mathrm{e}^{-sh}A_1 - \int_{-h}^{0} \mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right)^{-1}$$

is known as the characteristic matrix of the system.

The spectrum of the system consists of the zeros of the characteristic function

$$\Lambda = \{s \,|\, f(s) = 0\}.$$

If system (7.1) satisfies the Lyapunov condition (Definition 6.6), then the spectrum can be divided into two parts; the first one, $\Lambda^{(+)}$, includes eigenvalues with positive real part, the second one, $\Lambda^{(-)}$, includes eigenvalues with negative real part.

Remark 7.3. If the matrix D is Schur stable, then $\Lambda^{(+)}$ either is empty or contains at most a finite number of points. In the first case system (7.1) is exponentially stable.

Theorem 7.3. *Let the matrix D be Schur stable. If system (7.1) satisfies the Lyapunov condition, then the matrix*

$$\begin{aligned}\widetilde{U}(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\mathrm{e}^{-\tau\xi}\mathrm{d}\xi \\ + \sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{H^T(s)WH(-s)\mathrm{e}^{-\tau s}, s_0\right\} \\ + \sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{H^T(-s)WH(s)\mathrm{e}^{\tau s}, s_0\right\}\end{aligned} \tag{7.13}$$

is a Lyapunov matrix of the system associated with W.

Proof. The matrix D is Schur stable, and system (7.1) satisfies the Lyapunov condition. Thus, neither the matrix $H(s)$ nor $H(-s)$ has a pole on the imaginary axis of the complex plane. Let ξ be a real number; then for sufficiently large $|\xi|$ the matrix $H^T(i\xi)WH(-i\xi)\mathrm{e}^{-i\tau\xi}$ is of the order $|\xi|^{-2}$. This means that the improper integral on the right-hand side of (7.13) is well defined for all real τ.

Part 1: The proof of symmetry property (7.6) coincides with that of Theorem 3.5.
Part 2: We check that matrix (7.13) satisfies the algebraic property in the form (7.8). To this end, we compute the following matrix:

$$\begin{aligned}\mathcal{O} &= \widetilde{U}(0)A_0+\widetilde{U}(-h)A_1+\int_{-h}^{0}\widetilde{U}(\theta)G(\theta)\mathrm{d}\theta+A_0^T\widetilde{U}(0)+A_1^T\widetilde{U}(h)\\
&+\int_{-h}^{0}G^T(\theta)\widetilde{U}(-\theta)\mathrm{d}\theta\\
&-D^T\left[\widetilde{U}(h)A_0+\widetilde{U}(0)A_1+\int_{-h}^{0}\widetilde{U}(h+\theta)G(\theta)\mathrm{d}\theta\right]\\
&-\left[A_0^T\widetilde{U}(-h)+A_1^T\widetilde{U}(0)+\int_{-h}^{0}G^T(\theta)\widetilde{U}(-h-\theta)\mathrm{d}\theta\right]D.\end{aligned}$$

Observe first that

$$\begin{aligned}\mathcal{O}_1 &= \widetilde{U}(0)A_0+\widetilde{U}(-h)A_1+\int_{-h}^{0}\widetilde{U}(\theta)G(\theta)\mathrm{d}\theta\\
&=\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}H^T(\xi)WH(-\xi)\left[A_0+\mathrm{e}^{\xi h}A_1+\int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{d}\xi\\
&+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)WH(-s)\left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right],s_0\right\}\\
&+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)WH(s)\left[A_0+\mathrm{e}^{-sh}A_1+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right],s_0\right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}_2 &= A_0^T\widetilde{U}(0)+A_1^T\widetilde{U}(h)+\int_{-h}^{0}G^T(\theta)\widetilde{U}(-\theta)\mathrm{d}\theta\\
&=\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[A_0+\mathrm{e}^{-\xi h}A_1+\int_{-h}^{0}\mathrm{e}^{\xi\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(\xi)WH(-\xi)\mathrm{d}\xi\end{aligned}$$

$$+\sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{ \left[A_0+\mathrm{e}^{-sh}A_1+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(s)WH(-s), s_0\right\}$$

$$+\sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{ \left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(-s)WH(s), s_0\right\}.$$

Now

$$\begin{aligned}
\mathcal{O}_3 &= -D^T\left[\widetilde{U}(h)A_0+\widetilde{U}(0)A_1+\int_{-h}^{0}\widetilde{U}(h+\theta)G(\theta)\mathrm{d}\theta\right]\\
&= -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} D^T H^T(\xi)WH(-\xi)\mathrm{e}^{-\xi h}\left[A_0+\mathrm{e}^{\xi h}A_1+\int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{d}\xi\\
&\quad -\sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{ D^T H^T(s)WH(-s)\mathrm{e}^{-sh}\right.\\
&\quad \left.\times\left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right], s_0\right\}\\
&\quad -\sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{ D^T H^T(-s)WH(s)\mathrm{e}^{sh}\right.\\
&\quad \left.\times\left[A_0+\mathrm{e}^{-sh}A_1+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right], s_0\right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{O}_4 &= -\left[A_0^T\widetilde{U}(-h)+A_1^T\widetilde{U}(0)+\int_{-h}^{0}G^T(\theta)\widetilde{U}(-h-\theta)\mathrm{d}\theta\right]D\\
&= -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[A_0+\mathrm{e}^{-\xi h}A_1+\int_{-h}^{0}\mathrm{e}^{\xi\theta}G(\theta)\mathrm{d}\theta\right]^T\\
&\quad \times \mathrm{e}^{\xi h}H^T(\xi)WH(-\xi)D\mathrm{d}\xi\\
&\quad -\sum_{s_0\in\Lambda^{(+)}} \operatorname{Res}\left\{ \left[A_0+\mathrm{e}^{-sh}A_1+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]^T\right.
\end{aligned}$$

$$\times \, \mathrm{e}^{sh} H^T(s) W H(-s) D, s_0 \Bigg\}$$

$$- \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ \left[A_0 + \mathrm{e}^{sh} A_1 + \int_{-h}^{0} \mathrm{e}^{-s\theta} G(\theta) \mathrm{d}\theta \right]^T \right.$$

$$\left. \times \, \mathrm{e}^{-sh} H^T(-s) W H(s) D, s_0 \right\}.$$

It is a matter of simple calculation to check the identities

$$H(s) \left[A_0 + \mathrm{e}^{-sh} A_1 + \int_{-h}^{0} \mathrm{e}^{s\theta} G(\theta) \mathrm{d}\theta \right] = -I + sH(s) \left(I - \mathrm{e}^{-sh} D \right)$$

and

$$H(-s) \left[A_0 + \mathrm{e}^{sh} A_1 + \int_{-h}^{0} \mathrm{e}^{-s\theta} G(\theta) \mathrm{d}\theta \right] = -I - sH(-s) \left(I - \mathrm{e}^{sh} D \right).$$

Using these identities we present the matrices $\mathcal{O}_j$ as follows:

$$\mathcal{O}_1 = \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} \left[-H^T(\xi) W - \xi H^T(\xi) W H(-\xi) \left(I - \mathrm{e}^{\xi h} D \right) \right] \mathrm{d}\xi$$

$$+ \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ -H^T(s) W - sH^T(s) W H(-s) \left(I - \mathrm{e}^{sh} D \right), s_0 \right\}$$

$$+ \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ -H^T(-s) W + sH^T(-s) W H(s) \left(I - \mathrm{e}^{-sh} D \right), s_0 \right\},$$

$$\mathcal{O}_2 = \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} \left[-W H(-\xi) + \xi \left(I - \mathrm{e}^{-\xi h} D \right)^T H^T(\xi) W H(-\xi) \right] \mathrm{d}\xi$$

$$+ \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ -W H(-s) + s \left(I - \mathrm{e}^{-sh} D \right)^T H^T(s) W H(-s), s_0 \right\}$$

$$+ \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ -W H(s) - s \left(I - \mathrm{e}^{sh} D \right)^T H^T(-s) W H(s), s_0 \right\},$$

$$
\begin{aligned}
\mathcal{O}_3 = \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} & \Big\{ \mathrm{e}^{-\xi h} D^T H^T(\xi) W \\
& + \xi \mathrm{e}^{-\xi h} D^T H^T(\xi) W H(-\xi) \left(I - \mathrm{e}^{\xi h} D \right) \Big\} \mathrm{d}\xi \\
& + \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \Big\{ \mathrm{e}^{-sh} D^T H^T(s) W \\
& + s \mathrm{e}^{-sh} D^T H^T(s) W H(-s) \left(I - \mathrm{e}^{sh} D \right), s_0 \Big\} \\
& + \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \Big\{ \mathrm{e}^{sh} D^T H^T(-s) W \\
& - s \mathrm{e}^{sh} D^T H^T(-s) W H(s) \left(I - \mathrm{e}^{-sh} D \right), s_0 \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}_4 = \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} & \Big\{ \mathrm{e}^{\xi h} W H(-\xi) D \\
& - \xi \mathrm{e}^{\xi h} \left(I - \mathrm{e}^{-\xi h} D \right)^T H^T(\xi) W H(-\xi) D \Big\} \mathrm{d}\xi \\
& + \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \Big\{ \mathrm{e}^{sh} W H(-s) D \\
& - s \mathrm{e}^{sh} \left(I - \mathrm{e}^{-sh} D \right)^T H^T(s) W H(-s) D, s_0 \Big\} \\
& + \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \Big\{ \mathrm{e}^{-sh} W H(s) D \\
& + s \mathrm{e}^{-sh} \left(I - \mathrm{e}^{sh} D \right)^T H^T(-s) W H(s) D, s_0 \Big\}.
\end{aligned}
$$

Collecting similar terms we obtain that

$$
\begin{aligned}
\mathcal{O} = -\frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} & \left[\left(I - \mathrm{e}^{-\xi h} D \right)^T H^T(\xi) W + W H(-\xi) \left(I - \mathrm{e}^{\xi h} D \right) \right] \mathrm{d}\xi \\
& - \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ \left(I - \mathrm{e}^{-sh} D \right)^T H^T(s) W + W H(-s) \left(I - \mathrm{e}^{sh} D \right), s_0 \right\} \\
& - \sum_{s_0 \in \Lambda^{(+)}} \mathrm{Res} \left\{ \left(I - \mathrm{e}^{sh} D \right)^T H^T(-s) W + W H(s) \left(I - \mathrm{e}^{-sh} D \right), s_0 \right\}.
\end{aligned}
$$

If we take into account the equality

$$\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} WH(-\xi)\left(I-\mathrm{e}^{\xi h}D\right)\mathrm{d}\xi = \langle \lambda=-\xi\rangle$$
$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} WH(\lambda)\left(I-\mathrm{e}^{-\lambda h}D\right)\mathrm{d}\lambda$$

and the fact that the Lyapunov condition implies that the matrices

$$WH(-s)\left(I-\mathrm{e}^{sh}D\right),\left(I-\mathrm{e}^{sh}D\right)^{T}H^{T}(-s)W$$

have no poles in the set $\Lambda^{(+)}$, then

$$\mathcal{O} = -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[\left(I-\mathrm{e}^{-\xi h}D\right)^{T}H^{T}(\xi)W+WH(\xi)\left(I-\mathrm{e}^{-\xi h}D\right)\right]\mathrm{d}\xi \quad (7.14)$$
$$-\sum_{s_0\in\Lambda^{(+)}}\operatorname{Res}\left\{\left(I-\mathrm{e}^{-sh}D\right)^{T}H^{T}(s)W+WH(s)\left(I-\mathrm{e}^{-sh}D\right),s_0\right\}.$$

By the residue theorem

$$S_1 = \sum_{s_0\in\Lambda^{(+)}}\operatorname{Res}\left\{\left(I-\mathrm{e}^{-sh}D\right)^{T}H^{T}(s)W+WH(s)\left(I-\mathrm{e}^{-sh}D\right),s_0\right\}$$
$$= \lim_{r\to\infty}\frac{1}{2\pi i}\oint_{\Gamma(r)}\left[\left(I-\mathrm{e}^{-\xi h}D\right)^{T}H^{T}(\xi)W+WH(\xi)\left(I-\mathrm{e}^{-\xi h}D\right)\right]\mathrm{d}\xi,$$

where $\Gamma(r)$ is the Nyquist contour consisting of the semicircle $C(r) = \left\{re^{i\varphi}\,\middle|\,\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}$ and the segment $[ir,-ir]$ of the imaginary axis.
The contour integral is

$$J_1(r) = \frac{1}{2\pi i}\oint_{\Gamma(r)}\left[\left(I-\mathrm{e}^{-\xi h}D\right)^{T}H^{T}(\xi)W+WH(\xi)\left(I-\mathrm{e}^{-\xi h}D\right)\right]\mathrm{d}\xi$$
$$= -\frac{1}{2\pi i}\int_{-ir}^{ir}\left[\left(I-\mathrm{e}^{-\xi h}D\right)^{T}H^{T}(\xi)W+WH(\xi)\left(I-\mathrm{e}^{-\xi h}D\right)\right]\mathrm{d}\xi$$
$$+\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\left(I-\mathrm{e}^{-hr\mathrm{e}^{i\varphi}}D\right)^{T}H^{T}(re^{i\varphi})W+WH(re^{i\varphi})\left(I-\mathrm{e}^{-hr\mathrm{e}^{i\varphi}}D\right)\right]re^{i\varphi}\mathrm{d}\varphi.$$

Therefore,

$$\lim_{r\to\infty} J_1(r) = -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[\left(I-\mathrm{e}^{-\xi h}D\right)^T H^T(\xi)W + WH(\xi)\left(I-\mathrm{e}^{-\xi h}D\right)\right]\mathrm{d}\xi$$

$$+\lim_{r\to\infty}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\left(I-\mathrm{e}^{-hr\mathrm{e}^{i\varphi}}D\right)^T H^T(r\mathrm{e}^{i\varphi})W\right.$$

$$\left.+WH(r\mathrm{e}^{i\varphi})\left(I-\mathrm{e}^{-hr\mathrm{e}^{i\varphi}}D\right)\right]r\mathrm{e}^{i\varphi}\mathrm{d}\varphi.$$

Since $H(r\mathrm{e}^{i\varphi})\left(I-\mathrm{e}^{-hr\mathrm{e}^{i\varphi}}D\right)r\mathrm{e}^{i\varphi}\to I$, as $r\to\infty$, uniformly by $\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we conclude that

$$S_1 = -\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[\left(I-\mathrm{e}^{-\xi h}D\right)^T H^T(\xi)W + WH(\xi)\left(I-\mathrm{e}^{-\xi h}D\right)\right]\mathrm{d}\xi + W.$$

Comparing the preceding equality with (7.14) we conclude that $\mathcal{O} = -W$. Thus matrix (7.13) satisfies property (7.7).

Part 3: Let us address property (7.5). For a given $\tau > 0$ we compute the matrix

$$F(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\widetilde{U}(\tau)-\widetilde{U}(\tau-h)D\right]-\widetilde{U}(\tau)A_0-\widetilde{U}(\tau-h)A_1-\int_{-h}^{0}\widetilde{U}(\tau+\theta)G(\theta)\mathrm{d}\theta$$

$$=\frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}H^T(\xi)WH(-\xi)\left[-\xi I+\xi\mathrm{e}^{\xi h}D-A_0-\mathrm{e}^{\xi h}A_1\right.$$

$$\left.\times-\int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{-\tau\xi}\mathrm{d}\xi+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)WH(-s)\right.$$

$$\left.\times\left[-sI+s\mathrm{e}^{sh}D-A_0-\mathrm{e}^{sh}A_1-\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{-\tau s},s_0\right\}$$

$$+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)WH(s)\right.$$

$$\left.\times\left[sI-s\mathrm{e}^{-sh}D-A_0-\mathrm{e}^{-sh}A_1-\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{\tau s},s_0\right\}$$

$$= \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} H^T(\xi) W \mathrm{e}^{-\tau\xi} \mathrm{d}\xi$$

$$+ \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res}\left\{H^T(s) W \mathrm{e}^{-\tau s}, s_0\right\} + \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res}\left\{H^T(-s) W \mathrm{e}^{\tau s}, s_0\right\}.$$

Since the matrix $H(-s)$ has no poles in the set $\Lambda^{(+)}$, we have the sum

$$\sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res}\left\{H^T(-s) W \mathrm{e}^{\tau s}, s_0\right\} = 0_{n \times n},$$

and we obtain

$$F(\tau) = \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} H^T(\xi) W \mathrm{e}^{-\tau\xi} \mathrm{d}\xi + \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res}\left\{H^T(s) W \mathrm{e}^{-\tau s}, s_0\right\}.$$

Applying the residue theorem,

$$S_2 = \sum_{s_0 \in \Lambda^{(+)}} \operatorname{Res}\left\{H^T(s) W \mathrm{e}^{-\tau s}, s_0\right\} = \lim_{r \to \infty} \frac{1}{2\pi i} \oint_{\Gamma(r)} H^T(\xi) W \mathrm{e}^{-\tau\xi} \mathrm{d}\xi$$

$$= -\frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} H^T(\xi) W \mathrm{e}^{-\tau\xi} \mathrm{d}\xi + \lim_{r \to \infty} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H^T(r\mathrm{e}^{i\varphi}) W r\mathrm{e}^{i\varphi} \mathrm{e}^{-\tau r\mathrm{e}^{i\varphi}} \mathrm{d}\varphi.$$

By Jordan's theorem, the equality

$$\lim_{r \to \infty} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H^T(r\mathrm{e}^{i\varphi}) W r\mathrm{e}^{i\varphi} \mathrm{e}^{-\tau r\mathrm{e}^{i\varphi}} \mathrm{d}\varphi = 0_{n \times n}$$

holds for any $\tau > 0$, and we arrive at the conclusion that for $\tau > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\widetilde{U}(\tau) - \widetilde{U}(\tau - h) D\right] - \widetilde{U}(\tau) A_0 - \widetilde{U}(\tau - h) A_1 - \int_{-h}^{0} \widetilde{U}(\tau + \theta) G(\theta) \mathrm{d}\theta = 0_{n \times n}.$$

Since the preceding equality remains true as $\tau \to +0$, matrix (7.13) satisfies property (7.5) for $\tau \geq 0$. This concludes the proof. □

Corollary 7.1. *In the case where system (7.1) is exponentially stable the following expression for the Lyapunov matrix holds:*

$$U(\tau) = \frac{\text{V.P.}}{2\pi i} \int_{-i\infty}^{i\infty} H^T(\xi) W H(-\xi) \mathrm{e}^{-\tau\xi} \mathrm{d}\xi,$$

and the matrix is a unique solution of Eq. (7.5) that satisfies properties (7.6) and (7.7).

7.6 Computation Issue

In this section we address the computation of Lyapunov matrices. In general we must apply numerical schemes to compute an approximate Lyapunov matrix. But in some cases Lyapunov matrices can be found as solutions of a special boundary value problem for an auxiliary delay-free system of matrix differential equations.

7.6.1 A Particular Case

We begin with the case where the matrix $G(\theta)$ is a polynomial of the form

$$G(\theta) = \sum_{j=1}^{m} \theta^{j-1} B_j. \tag{7.15}$$

Here $B_1, \ldots, B_m$ are given constant $n \times n$ matrices.

Let $U(\tau)$ be a Lyapunov matrix of system (7.1). We define for $\tau \in [0,h]$ the following set of $2(m+1)$ auxiliary matrices:

$$\begin{cases} Z(\tau) = U(\tau), \quad X_j(\tau) = \int_{-h}^{0} \theta^{j-1} U(\tau+\theta) \mathrm{d}\theta, \quad j = 1, \ldots, m, \\ V(\tau) = U(\tau-h), \quad Y_j(\tau) = \int_{-h}^{0} \theta^{j-1} U(\tau-\theta-h) \mathrm{d}\theta, \quad j = 1, \ldots, m. \end{cases} \tag{7.16}$$

Property (7.5) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[Z(\tau) - V(\tau)D] = Z(\tau)A_0 + V(\tau)A_1 + \sum_{j=1}^{m} X_j(\tau) B_j.$$

At the same time,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^TZ(\tau)+V(\tau)\right] &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(h-\tau)-U(-\tau)D\right]^T \\ &= -\left[U(h-\tau)A_0-U(-\tau)A_1\right. \\ &\quad \left. -\sum_{j=1}^{m}\left(\int_{-h}^{0}\theta^{j-1}U(h-\tau+\theta)\mathrm{d}\theta\right)B_j\right]^T.\end{aligned}$$

Observe that

$$U^T(h-\tau)=V(\tau), \quad U^T(-\tau)=Z(\tau)$$

and

$$\left[\int_{-h}^{0}\theta^{j-1}U(h-\tau+\theta)\mathrm{d}\theta\right]^T = X_j^T(h-\tau)=Y_j(\tau), \quad j=1,\dots,m.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^TZ(\tau)+V(\tau)\right] = -A_0^TV(\tau)-A_1^TZ(\tau)-\sum_{j=1}^{m}B_j^TY_j(\tau).$$

The first derivative of the auxiliary matrices $X_1(\tau)$ and $Y_1(\tau)$ is

$$\frac{\mathrm{d}X_1(\tau)}{\mathrm{d}\tau}=\frac{\mathrm{d}Y_1(\tau)}{\mathrm{d}\tau}=Z(\tau)-V(\tau).$$

Now

$$\begin{aligned}\frac{\mathrm{d}X_j(\tau)}{\mathrm{d}\tau} &= -(-h)^{j-1}U(\tau-h)-(j-1)\int_{-h}^{0}\theta^{j-2}U(\tau+\theta)\mathrm{d}\theta \\ &= -(-h)^{j-1}V(\tau)-(j-1)X_{j-1}(\tau), \quad j=2,\dots,m,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}Y_j(\tau)}{\mathrm{d}\tau} &= (-h)^{j-1}U(\tau)+(j-1)\int_{-h}^{0}\theta^{j-2}U(\tau-\theta-h)\mathrm{d}\theta \\ &= (-h)^{j-1}Z(\tau)+(j-1)Y_{j-1}(\tau), \quad j=2,\dots,m.\end{aligned}$$

And we arrive at the following system of delay-free matrix differential equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left[Z(\tau)-V(\tau)D\right]=Z(\tau)A_0+V(\tau)A_1+\sum_{j=1}^{m}X_j(\tau)B_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^TZ(\tau)+V(\tau)\right]=-A_1^TZ(\tau)-A_0^TV(\tau)-\sum_{j=1}^{m}B_j^TY_j(\tau), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}X_1(\tau)=Z(\tau)-V(\tau), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}Y_1(\tau)=Z(\tau)-V(\tau), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}X_j(\tau)=-(-h)^{j-1}V(\tau)-(j-1)X_{j-1}(\tau), \quad j=2,\dots,m, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}Y_j(\tau)=(-h)^{j-1}Z(\tau)+(j-1)Y_{j-1}(\tau), \quad j=2,\dots,m. \end{cases} \tag{7.17}$$

The auxiliary matrices also satisfy some boundary value conditions: It follows from (7.16) that

$$Z(0)=V^T(h)$$

and

$$\begin{aligned} X_j(h) &= \int_{-h}^{0}\theta^{j-1}U(h+\theta)\mathrm{d}\theta=\left[\int_{-h}^{0}\theta^{j-1}U(-\theta-h)\mathrm{d}\theta\right]^T \\ &= Y_j^T(0), \quad j=1,\dots,m, \\ Y_j(h) &= \int_{-h}^{0}\theta^{j-1}U(h-\theta-h)\mathrm{d}\theta=\left[\int_{-h}^{0}\theta^{j-1}U(\theta)\mathrm{d}\theta\right]^T \\ &= X_j^T(0), \quad j=1,\dots,m. \end{aligned}$$

The algebraic property written in the terms of the auxiliary matrices has the form

$$\begin{aligned} -W = {} & Z(0)A_0+V(0)A_1+\sum_{j=1}^{m}X_j(0)B_j \\ & -D^T\left[Z(h)A_0+V(h)A_1+\sum_{j=1}^{m}X_j(h)B_j\right] \end{aligned}$$

$$+A_0^T V(h) + A_1^T Z(h) A_1 + \sum_{j=1}^{m} B_j^T Y_j(h)$$
$$-\left[A_0^T V(0) + A_1^T Z(0) + \sum_{j=1}^{m} B_j^T Y_j(0)\right] D.$$

We summarize the results of our analysis in the following statement.

Theorem 7.4. *Given a time-delay system (7.1), where the matrix $G(\theta)$ is of the form (7.15), let $U(\tau)$ be a Lyapunov matrix of the system associated with a symmetric matrix W. Then there exists a solution*

$$\{Z(\tau), V(\tau), X_1(\tau), \dots, X_m(\tau), Y_1(\tau), \dots, Y_m(\tau)\}$$

of the delay-free system (7.17) such that $Z(\tau) = U(\tau)$, $\tau \in [0,h]$. The solution satisfies the following set of boundary value conditions:

$$\begin{cases} Z(0) = V^T(h), \\ X_j(h) = Y_j^T(0), \text{ and } Y_j(h) = X_j^T(0), \quad j = 1, \dots, m, \\ -W = Z(0)A_0 + V(0)A_1 + \sum_{j=1}^{m} X_j(0) B_j \\ \qquad -D^T \left[Z(h)A_0 + V(h)A_1 + \sum_{j=1}^{m} X_j(h) B_j\right] \\ \qquad +A_0^T V(h) + A_1^T Z(h) A_1 + \sum_{j=1}^{m} B_j^T Y_j(h) \\ \qquad -\left[A_0^T V(0) + A_1^T Z(0) + \sum_{j=1}^{m} B_j^T Y_j(0)\right] D. \end{cases} \tag{7.18}$$

Remark 7.4. System (7.17) is regular if and only if the spectrum of the matrix D does not contain a point λ_0 such that λ_0^{-1} also belongs to the spectrum.

The following lemma demonstrates that some relations exist between the auxiliary matrices.

Lemma 7.2. *The auxiliary matrices $X_k(\tau)$ and $Y_k(\tau)$, $k = 1, \dots, m$, satisfy the relations*

$$X_k(\tau) = (-1)^{k-1} \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!} h^j Y_{k-j}(\tau), \quad k = 1, \dots, m,$$

and

$$Y_k(\tau) = (-1)^{k-1}\sum_{j=0}^{k-1}\frac{(k-1)!}{j!(k-1-j)!}h^j X_{k-j}(\tau), \quad k=1,\dots,m.$$

Proof. The first set of relations can be easily obtained as follows. By definition,

$$\begin{aligned} X_k(\tau) &= \int_{-h}^{0}\theta^{k-1}U(\tau+\theta+h-h)\mathrm{d}\theta = \langle \xi = -\theta - h\rangle \\ &= \int_{-h}^{0}(-h-\xi)^{k-1}U(\tau-\xi-h)\mathrm{d}\xi \\ &= (-1)^{k-1}\sum_{j=0}^{k-1}\frac{(k-1)!}{j!(k-1-j)!}h^j Y_{k-j}(\tau). \end{aligned}$$

The second set of relations can be verified in a similar way. □

Lemma 7.2 provides a substantial reduction of system (7.17). Observe that the sum

$$\sum_{j=1}^{m}B_j^T Y_j(\tau) = \sum_{j=1}^{m}(-1)^{j-1}B_j^T\left(\sum_{\nu=0}^{j-1}\frac{(j-1)!}{\nu!(j-1-\nu)!}h^{\nu-1}X_{j-\nu}(\tau)\right).$$

Thus, if we define the matrix

$$B(\xi) = \sum_{j=1}^{m}(-\xi)^{j-1}B_j^T,$$

then

$$\sum_{j=1}^{m}B_j^T Y_j(\tau) = \sum_{k=1}^{m}\frac{1}{(k-1)!}B^{(k-1)}(h)X_k(\tau),$$

and the second equation of system (7.17) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T Z(\tau)+V(\tau)\right] = -A_1^T Z(\tau) - A_0^T V(\tau) - \sum_{k=1}^{m}\frac{1}{(k-1)!}B^{(k-1)}(h)X_k(\tau).$$

Therefore, system (7.17) is reduced to the following set of $(m+2)$ matrix equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[Z(\tau)-V(\tau)D\right]=Z(\tau)A_0+V(\tau)A_1+\sum_{j=1}^{m}X_j(\tau)B_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left[V(\tau)-D^TZ(\tau)\right]=-A_1^TZ(\tau)-A_0^TV(\tau)-\sum_{k=1}^{m}\frac{1}{(k-1)!}B^{(k-1)}(h)X_k(\tau), \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_1(\tau)=Z(\tau)-V(\tau), \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_j(\tau)=-(-h)^jV(\tau)-jX_{j-1}(\tau), \quad j=2,\dots,m. \end{cases} \tag{7.19}$$

The set of boundary value conditions (7.18) is now of the form

$$\begin{cases} Z(0)=V(h), \\ X_k^T(0)=(-1)^{k-1}\sum_{j=0}^{k-1}\frac{(k-1)!}{j!(k-j-1)!}h^jX_{k-j}(h), \quad k=1,\dots,m, \\ -W=Z(0)A_0+V(0)A_1+\sum_{j=1}^{m}X_j(0)B_j \\ \qquad -D^T\left[Z(h)A_0+V(h)A_1+\sum_{j=1}^{m}X_j(h)B_j\right] \\ \qquad +A_0^TV(h)+A_1^TZ(h)A_1+\sum_{j=1}^{m}B_j^TX_j^T(0) \\ \qquad -\left[A_0^TV(0)+A_1^TZ(0)+\sum_{j=1}^{m}B_j^TX_j^T(h)\right]D. \end{cases} \tag{7.20}$$

There is a certain connection between the spectrum of time-delay system (7.1) and that of delay-free system (7.17).

Theorem 7.5. *Given a time-delay system (7.1), where the matrix $G(\theta)$ is of the form (7.15), let s_0 be an eigenvalue of the system such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of delay-free system (7.17).*

Proof. The characteristic matrix of system (7.1) is of the form

$$G(s)=sI-s\mathrm{e}^{-hs}D-A_0-\mathrm{e}^{-hs}A_1-\sum_{k=1}^{m}f^{(k-1)}(s)B_k,$$

where

$$f^{(0)}(s)=\frac{1-\mathrm{e}^{-hs}}{s}, \text{ and } f^{(j)}(s)=\frac{\mathrm{d}^jf^{(0)}(s)}{\mathrm{d}s^j}, \quad j=1,2,\dots,m-1.$$

Because s_0 and $-s_0$ are eigenvalues of the system, there exist two nontrivial vectors γ and μ such that

$$\gamma^T G(s_0) = 0, \quad G^T(-s_0)\mu = 0. \tag{7.21}$$

On the other hand, a complex number s belongs to the spectrum of delay-free system (7.17) if and only if there exists a nontrivial set of $n \times n$ constant matrices

$$\left\{ Z^{(0)}, V^{(0)}, X_1^{(0)}, \ldots, X_m^{(0)}, Y_1^{(0)}, \ldots, Y_m^{(0)} \right\}$$

such that

$$\begin{cases} sZ^{(0)} - sV^{(0)}D = Z^{(0)}A_0 + V^{(0)}A_1 + \sum_{j=1}^{m} X_j^{(0)} B_j, \\ -sD^T Z^{(0)} + sV^{(0)} = -A_1^T Z^{(0)} - A_0^T V^{(0)} - \sum_{j=1}^{m} B_j^T Y_j^{(0)}, \\ sX_1^{(0)} = Z^{(0)} - V^{(0)}, \\ sY_1^{(0)} = Z^{(0)} - V^{(0)}, \\ sX_j^{(0)} = -(-h)^{j-1} V^{(0)} - (j-1) X_{j-1}^{(0)}, \quad j = 2, \ldots, m, \\ sY_j^{(0)} = (-h)^{j-1} Z^{(0)} + (j-1) Y_{j-1}^{(0)}, \quad j = 2, \ldots, m. \end{cases} \tag{7.22}$$

Multiplying the first equality in (7.21) by μ from the left-hand side and the second equality by $-\mathrm{e}^{-hs_0}\gamma^T$ from the right-hand side we obtain

$$0_{n\times n} = s_0 \mu\gamma^T - s_0 \mathrm{e}^{-hs_0} \mu\gamma^T D - \mu\gamma^T A_0 - \mathrm{e}^{-hs_0} \mu\gamma^T A_1 - \sum_{k=1}^{m} f^{(k-1)}(s_0) \mu\gamma^T B_k$$

and

$$\begin{aligned} 0_{n\times n} = {} & s_0 \mathrm{e}^{-hs_0} \mu\gamma^T - s_0 D^T \mu\gamma^T + A_0^T \mathrm{e}^{-hs_0} \mu\gamma^T + A_1^T \mu\gamma^T \\ & + \sum_{k=1}^{m} \mathrm{e}^{-hs_0} f^{(k-1)}(-s_0) B_k^T \mu\gamma^T. \end{aligned}$$

Let us introduce the matrices

$$Z^{(0)} = \mu\gamma^T, \quad V^{(0)} = \mathrm{e}^{-hs_0} \mu\gamma^T$$

and

$$X_j^{(0)} = f^{(j-1)}(s_0) \mu\gamma^T, Y_j^{(0)} = \mathrm{e}^{-hs_0} f^{(j-1)}(-s_0) \mu\gamma^T, \quad j = 1, \ldots, m.$$

Then the preceding equalities take the form

$$s_0Z^{(0)} - s_0V^{(0)}D - Z^{(0)}A_0 - V^{(0)}A_1 - \sum_{k=1}^{m} X_k^{(0)}B_k = 0_{n\times n},$$

$$-s_0D^TZ^{(0)} + s_0V^{(0)} + A_0^TV^{(0)} + A_1^TZ^{(0)} + \sum_{k=1}^{m} B_k^TY_k^{(0)} = 0_{n\times n}.$$

That is, for $s = s_0$ the matrices satisfy the first two equations of system (7.22). It is a matter of simple calculation to show that for $s = s_0$ the matrices also satisfy the remaining $2m$ matrix equations in (7.22). It is evident that the introduced set of matrices

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \ldots, X_m^{(0)}, Y_1^{(0)}, \ldots, Y_m^{(0)}\right\}$$

is not trivial. Therefore, the complex value s_0 belongs to the spectrum of system (7.17). The same is true also for $-s_0$. □

Remark 7.5. The statement of Theorem 7.5 remains valid if we replace system (7.17) by the reduced system (7.19).

7.6.2 A Special Case

Now we consider the case where the matrix $G(\theta)$ is of the form

$$G(\theta) = \sum_{j=1}^{m} \eta_j(\theta)B_j, \tag{7.23}$$

where $B_1, \ldots, B_m$ are given $n \times n$ matrices and the scalar functions $\eta_1(\theta), \ldots, \eta_m(\theta)$ are such that

$$\frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} = \sum_{k=1}^{m} \alpha_{jk}\eta_k(\theta), \quad j = 1, \ldots, m. \tag{7.24}$$

Remark 7.6. In the previous subsection $\eta_j(\theta) = \theta^{j-1}$, $j = 1, \ldots, m$. These functions satisfy the equations

$$\frac{\mathrm{d}\eta_1(\theta)}{\mathrm{d}\theta} = 0, \quad \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} = (j-1)\,\eta_{j-1}(\theta), \quad j = 2, \ldots, m.$$

Matrix equation (7.5) for $U(\tau)$ is now of the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(\tau)-U(\tau-h)D\right] = U(\tau)A_0 + U(\tau-h)A_1 + \sum_{j=1}^{m}\left(\int_{-h}^{0}\eta_j(\theta)U(\tau+\theta)\mathrm{d}\theta\right)B_j, \quad \tau \geq 0. \tag{7.25}$$

Once again, we introduce the matrices $Z(\tau) = U(\tau)$, $V(\tau) = U(\tau - h)$, and

$$X_j(\tau) = \int_{-h}^{0}\eta_j(\theta)U(\tau+\theta)\mathrm{d}\theta, \quad Y_j(\tau) = \int_{-h}^{0}\eta_j(\theta)U(\tau-\theta-h)\mathrm{d}\theta, \quad j = 1,\ldots,m.$$

Then Eq. (7.25) has the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[Z(\tau)-V(\tau)D\right] = Z(\tau)A_0 + V(\tau)A_1 + \sum_{j=1}^{m}X_j(\tau)B_j, \quad \tau \in [0,h]$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^T Z(\tau) + V(\tau)\right] = -A_1^T Z(\tau) - A_0^T V(\tau) - \sum_{j=1}^{m} B_j^T Y_j(\tau).$$

Now,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}X_j(\tau) &= \eta_j(0)Z(\tau) - \eta_j(-h)V(\tau) - \int_{-h}^{0}\frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta}U(\tau+\theta)\mathrm{d}\theta \\ &= \eta_j(0)Z(\tau) - \eta_j(-h)V(\tau) - \sum_{k=1}^{m}\alpha_{jk}X_k(\tau), \quad j = 1,\ldots,m,\end{aligned}$$

whereas

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}Y_j(\tau) &= -\eta_j(0)V(\tau) + \eta_j(-h)Z(\tau) + \int_{-h}^{0}\frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta}U(\tau-\theta-h)\mathrm{d}\theta \\ &= \eta_j(-h)Z(\tau) - \eta_j(0)V(\tau) + \sum_{k=1}^{m}\alpha_{jk}Y_k(\tau), \quad j = 1,\ldots,m.\end{aligned}$$

And we arrive at the following system of delay-free matrix equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}[Z(\tau)-V(\tau)D] = Z(\tau)A_0+V(\tau)A_1+\sum_{j=1}^{m}X_j(\tau)B_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-D^TZ(\tau)+V(\tau)\right] = -A_0^TV(\tau)-A_1^TZ(\tau)-\sum_{j=1}^{m}B_j^TY_j(\tau), \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_j(\tau) = \eta_j(0)Z(\tau)-\eta_j(-h)V(\tau)-\sum_{k=1}^{m}\alpha_{jk}X_k(\tau), \quad j=1,\dots,m, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}Y_j(\tau) = \eta_j(-h)Z(\tau)-\eta_j(0)V(\tau)+\sum_{k=1}^{m}\alpha_{jk}Y_k(\tau), \quad j=1,\dots,m. \end{cases} \tag{7.26}$$

As in the previous case, we obtain the following result.

Theorem 7.6. *Given a time-delay system (7.1), where the matrix $G(\theta)$ is of the form (7.23), let $U(\tau)$ be a Lyapunov matrix of the system associated with a symmetric matrix W. There exists a solution,*

$$\{Z(\tau),V(\tau),X_1(\tau),\dots,X_m(\tau),Y_1(\tau),\dots,Y_m(\tau)\},$$

of the delay-free system of matrix Eqs. (7.26) such that $U(\tau)=Z(\tau)$, $\tau\in[0,h]$. The solution satisfies boundary value conditions (7.18).

The statement of Theorem 7.5 remains true for this new case.

Theorem 7.7. *Given a time-delay system (7.1), where the matrix $G(\theta)$ is of the form (7.23), let s_0 be an eigenvalue of the system such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of delay-free system (7.26).*

Let functions $\eta_j(\theta)$, $j=1,\dots,m$, satisfy, for $\theta\in[-h,0]$, the equalities

$$\eta_j(-\theta-h)=\sum_{k=1}^{m}\gamma_{jk}\eta_k(\theta), \quad j=1,\dots,m,$$

with constant coefficients γ_{jk}. Then

$$\begin{aligned} Y_j(\tau) &= \int_{-h}^{0}\eta_j(\theta)U(\tau-\theta-h)\mathrm{d}\theta = \langle \xi=-\theta-h\rangle \\ &= \int_{-h}^{0}\eta_j(-\xi-h)U(\tau+\xi)\mathrm{d}\xi \\ &= \sum_{k=1}^{m}\gamma_{jk}\int_{-h}^{0}\eta_k(\xi)U(\tau+\xi)\mathrm{d}\xi = \sum_{k=1}^{m}\gamma_{jk}X_k(\tau), \end{aligned}$$

and one can exclude the matrices $Y_j(\tau)$ of system (7.26) and boundary value conditions (7.18).

7.7 A New Form of Lyapunov Functionals

Here we present functional (7.3) in a new form that does not include terms that depend on φ'. To this end, we assume here that $\varphi \in C^1([-h,0],R^n)$. We will transform three terms in (7.3).

The first one is the term

$$\begin{aligned} J_1 &= 2\varphi^T(0)\int_{-h}^{0}\left[U(-h-\theta)-D^TU(-\theta)\right][A_1\varphi(\theta)+D\varphi'(\theta)]\mathrm{d}\theta \\ &= 2\varphi^T(0)\int_{-h}^{0}\left[U(-h-\theta)-D^TU(-\theta)\right]A_1\varphi(\theta)\mathrm{d}\theta \\ &\quad + 2\varphi^T(0)\left[U(-h)-D^TU(0)\right]D\varphi(0) - 2\varphi^T(0)\left[U(0)-D^TU(h)\right]D\varphi(-h) \\ &\quad + 2\varphi^T(0)\int_{-h}^{0}\left[U'(-h-\theta)-D^TU'(-\theta)\right]D\varphi(\theta)\mathrm{d}\theta. \end{aligned}$$

The second term is

$$\begin{aligned} J_2 &= \int_{-h}^{0}\left[A_1\varphi(\theta_1)+D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)\left[A_1\varphi(\theta_2)+D\varphi'(\theta_2)\right]\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\ &= \int_{-h}^{0}\varphi^T(\theta_1)\left(\int_{-h}^{0}A_1^TU(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\ &\quad + 2\int_{-h}^{0}[A_1\varphi(\theta_1)]^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)D\varphi'(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\ &\quad + \int_{-h}^{0}\left[D\varphi'(\theta_1)\right]^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)D\varphi'(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1. \end{aligned}$$

Since the integral

$$\begin{aligned} J &= \int_{-h}^{0}U(\theta_1-\theta_2)D\varphi'(\theta_2)\mathrm{d}\theta_2 \\ &= U(\theta_1)D\varphi(0)-U(\theta_1+h)D\varphi(-h)+\int_{-h}^{0}U'(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2, \end{aligned}$$

then

$$J_2 = \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} \left[A_1^T U(\theta_1-\theta_2) A_1 \varphi(\theta_2) \mathrm{d}\theta_2 + 2A_1^T U'(\theta_1-\theta_2) D \right] \right) \mathrm{d}\theta_1$$

$$- \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} D^T U''(\theta_1-\theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

$$+ 2\varphi^T(0) D^T \int_{-h}^{0} \left[U(-\theta)A_1 + U'(-\theta)D \right] \varphi(\theta) \mathrm{d}\theta$$

$$- 2\varphi^T(-h) D^T \int_{-h}^{0} \left[U(-\theta-h)A_1 + U'(-\theta-h)D \right] \varphi(\theta) \mathrm{d}\theta$$

$$+ \varphi^T(0) D^T U(0) D\varphi(0) - 2\varphi^T(0) D^T U(h) D\varphi(-h)$$

$$+ \varphi^T(-h) D^T U(0) D\varphi(-h).$$

Remark 7.7. In the computation of the term

$$Q = -\int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} D^T U''(\theta_1-\theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

one must remember that the first derivative of the Lyapunov matrix, $U'(\tau)$, suffers a jump discontinuity at the point $\tau = 0$; see Eq. (7.7). Therefore, this term can be presented as

$$Q = -\int_{-h}^{0} \varphi^T(\theta) D^T \left[U'(+0) - U'(-0) \right] D\varphi(\theta) \mathrm{d}\theta$$

$$- \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{\theta_1-0} D^T U''(\theta_1-\theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right.$$

$$\left. + \int_{\theta_1+0}^{0} D^T U''(\theta_1-\theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1.$$

Finally, we consider the term

$$
\begin{aligned}
J_3 = {} & 2\int_{-h}^{0} \left[A_1\varphi(\theta_1)+D\varphi'(\theta_1)\right]^T \\
& \times \left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2} U(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\
= {} & 2\int_{-h}^{0}\varphi^T(\theta_1)\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2} A_1^T U(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\
& + 2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta} D^T U(h-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
& - 2\varphi^T(-h)\int_{-h}^{0}\left(\int_{-h}^{\theta} D^T U(-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
& - 2\int_{-h}^{0}\varphi^T(\theta_1)\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2} D^T U'(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1.
\end{aligned}
$$

Now we substitute into (7.3) these new expressions and collect similar terms. We start with the terms that do not include an integral factor. The sum of the terms is

$$
\begin{aligned}
S_1 = {} & \varphi^T(0)\left[U(0)-U(-h)D-D^TU(h)+D^TU(0)D\right]\varphi(0) \\
& + 2\varphi^T(0)\left[U(-h)-D^TU(0)\right]D\varphi(0) - 2\varphi^T(0)\left[U(0)-D^TU(h)\right]D\varphi(-h) \\
& + \varphi^T(0)D^TU(0)D\varphi(0) - \varphi^T(0)D^TU(h)D\varphi(-h) \\
& - \varphi^T(-h)D^TU(-h)D\varphi(0) + \varphi^T(-h)D^TU(0)D\varphi(-h) \\
= {} & \left[\varphi(0)-D\varphi(-h)\right]^T U(0)\left[\varphi(0)-D\varphi(-h)\right].
\end{aligned}
$$

Then we collect terms that include the factor $\varphi^T(0)$ and an integral factor

$$
\begin{aligned}
S_2 = {} & 2\varphi^T(0)\int_{-h}^{0}\left[U(-h-\theta)-D^TU(-\theta)\right]A_1\varphi(\theta)\mathrm{d}\theta \\
& + 2\varphi^T(0)\int_{-h}^{0}\left[U'(-h-\theta)-D^TU'(-\theta)\right]D\varphi(\theta)\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\varphi^T(0)\int_{-h}^{0} D^T U(-\theta)A_1\varphi(\theta)\mathrm{d}\theta + 2\varphi^T(0)\int_{-h}^{0} D^T U'(-\theta)D\varphi(\theta)\mathrm{d}\theta \\
&+ 2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta} D^T U(h-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
&+ 2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta}\left[U(-\theta+\xi) - D^T U(-\theta+\xi+h)\right]G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
= {}& 2\varphi^T(0)\int_{-h}^{0}\left(U(-h-\theta)A_1 + U'(-h-\theta)D\right. \\
&\left. + \int_{-h}^{\theta} U(-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
= {}& 2\varphi^T(0)\int_{-h}^{0}\left(U'(-\theta) - U(-\theta)A_0 - \int_{\theta}^{0} U(-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta.
\end{aligned}
$$

Now we collect terms that include the factor $\varphi^T(-h)$ and an integral factor

$$
\begin{aligned}
S_3 = {}& -2\varphi^T(-h)\int_{-h}^{0} D^T U(-\theta-h)A_1\varphi(\theta)\mathrm{d}\theta \\
& -2\varphi^T(-h)\int_{-h}^{0} D^T U'(-\theta-h)D\varphi(\theta)\mathrm{d}\theta \\
& -2\varphi^T(-h)\int_{-h}^{0}\left(\int_{-h}^{\theta} D^T U(-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta \\
= {}& -2\left[D\varphi(-h)\right]^T\int_{-h}^{0}\left(U(-\theta-h)A_1 + U'(-\theta-h)D\right. \\
&\left. + \int_{-h}^{\theta} U(-\theta+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta
\end{aligned}
$$

$$= -2\left[D\varphi(-h)\right]^T \int_{-h}^{0} \left(U'(-\theta) - U(-\theta)A_0 - \int_{\theta}^{0} U(-\theta+\xi)G(\xi)\mathrm{d}\xi \right) \varphi(\theta)\mathrm{d}\theta.$$

Thus,

$$S_2 + S_3 = 2\left[\varphi(0) - D\varphi(-h)\right]^T \int_{-h}^{0} \left(U'(-\theta) - U(-\theta)A_0 \right.$$
$$\left. - \int_{\theta}^{0} U(-\theta+\xi)G(\xi)\mathrm{d}\xi \right) \varphi(\theta)\mathrm{d}\theta.$$

We arrive at the desired new form of the functional

$$\begin{aligned} v_0(\varphi) = {} & \left[\varphi(0) - D\varphi(-h)\right]^T U(0) \left[\varphi(0) - D\varphi(-h)\right] \\ & + 2\left[\varphi(0) - D\varphi(-h)\right]^T \int_{-h}^{0} \left[U(-h-\theta)A_1 + U'(-h-\theta)D \right. \\ & \left. + \int_{-h}^{\theta} U(-\theta+\xi)G(\xi)\mathrm{d}\xi \right] \varphi(\theta)\mathrm{d}\theta \\ & - \int_{-h}^{0} \varphi^T(\theta_1) D^T \left(\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \right. \\ & \left. + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\ & - \int_{-h}^{0} \varphi^T(\theta) D^T \left[U'(+0) - U'(-0) \right] D\varphi(\theta)\mathrm{d}\theta \\ & + \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} \left[A_1^T U(\theta_1 - \theta_2) A_1 + A_1^T U'(\theta_1 - \theta_2)\, D \right. \right. \\ & \left. \left. - D^T U'(\theta_1 - \theta_2) A_1 \right] \varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\ & + \int_{-h}^{0} \varphi^T(\theta_1) \left\{ \int_{-h}^{0} \left(\int_{-h}^{\theta_1} G^T\,(\xi_1) \right. \right. \end{aligned}$$

$$\times \left[\int_{-h}^{\theta_2} U(\theta_1 - \xi_1 - \theta_2 + \xi_2) G(\xi_2) \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \Bigg) \varphi(\theta_2) \mathrm{d}\theta_2 \Bigg\} \mathrm{d}\theta_1$$

$$+2 \int_{-h}^{0} \varphi^T(\theta_1) \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} A_1^T U(\theta_1 + h - \theta_2 + \xi) G(\xi) \mathrm{d}\xi \right) \varphi(\theta_2) \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

$$-2 \int_{-h}^{0} \varphi^T(\theta_1) \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} D^T U'(\theta_1 + h - \theta_2 + \xi) G(\xi) \mathrm{d}\xi \right) \varphi(\theta_2) \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1. \tag{7.27}$$

7.8 Complete Type Functionals

For the given symmetric matrices W_j, $j = 0, 1, 2$, we define the functional

$$w(\varphi) = \varphi^T(0) W_0 \varphi(0) + \varphi^T(-h) W_1 \varphi(-h) + \int_{-h}^{0} \varphi^T(\theta) W_2 \varphi(\theta) \mathrm{d}\theta, \quad \varphi \in PC^1([-h, 0], R^n).$$

Theorem 7.8. *Let $U(\tau)$ be a Lyapunov matrix of system (7.1) associated with the matrix*

$$W = W_0 + W_1 + hW_2.$$

Then the functional

$$v(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta) \left[W_1 + (h + \theta) W_2 \right] \varphi(\theta) \mathrm{d}\theta, \quad \varphi \in PC^1([-h, 0], R^n), \tag{7.28}$$

where $v_0(\varphi)$ is defined by (7.3) with this Lyapunov matrix, is such that

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) = -w(x_t), \quad t \geq 0,$$

along the solutions of the system.

Definition 7.2. We say that functional (7.28) is of the complete type if the matrices W_j, $j = 0, 1, 2$, are positive definite.

In explicit form functional (7.28) has the form

$$
\begin{aligned}
v(\varphi) = {} & [\varphi(0) - D\varphi(-h)]^T U(0) [\varphi(0) - D\varphi(-h)] \\
& + 2[\varphi(0) - D\varphi(-h)]^T \int_{-h}^{0} \Big[U(-h-\theta)A_1 + U'(-h-\theta)D \\
& + \int_{-h}^{\theta} U(-\theta+\xi)G(\xi)\mathrm{d}\xi \Big] \varphi(\theta)\mathrm{d}\theta \\
& - \int_{-h}^{0} \varphi^T(\theta_1) D^T \Bigg(\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \\
& + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2 \Bigg) \mathrm{d}\theta_1 \\
& + \int_{-h}^{0} \varphi^T(\theta) \left(W_1 + (h+\theta)W_2 - D^T \left[U'(+0) - U'(-0) \right] D \right) \varphi(\theta)\mathrm{d}\theta \\
& + \int_{-h}^{0} \varphi^T(\theta_1) \Bigg(\int_{-h}^{0} \left[A_1^T U(\theta_1 - \theta_2) A_1 + A_1^T U'(\theta_1 - \theta_2) D \right. \\
& \left. - D^T U'(\theta_1 - \theta_2) A_1 \right] \varphi(\theta_2)\mathrm{d}\theta_2 \Bigg) \mathrm{d}\theta_1 \\
& + \int_{-h}^{0} \varphi^T(\theta_1) \Bigg\{ \int_{-h}^{0} \\
& \left(\int_{-h}^{\theta_1} G^T(\xi_1) \left[\int_{-h}^{\theta_2} U(\theta_1 - \xi_1 - \theta_2 + \xi_2) G(\xi_2)\mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \right) \varphi(\theta_2)\mathrm{d}\theta_2 \Bigg\} \mathrm{d}\theta_1 \\
& + 2\int_{-h}^{0} \varphi^T(\theta_1) \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} A_1^T U(\theta_1 + h - \theta_2 + \xi) G(\xi)\mathrm{d}\xi \right) \varphi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 \\
& - 2\int_{-h}^{0} \varphi^T(\theta_1) \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} D^T U'(\theta_1 + h - \theta_2 + \xi) G(\xi)\mathrm{d}\xi \right) \varphi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 .
\end{aligned}
\tag{7.29}
$$

7.9 Quadratic Bounds

Lemma 7.3. *Let system (7.1) be exponentially stable. If the matrices W_j, $j=0,1,2$, are positive definite, then there exists $\alpha_1>0$ such that the complete type functional (7.28) satisfies the inequality*

$$\alpha_1 \left\|\varphi(0)-D\varphi(-h)\right\|^2 \leq v(\varphi), \quad \varphi \in PC^1([-h,0],R^n).$$

Proof. Consider the functional

$$\widetilde{v}(\varphi)=v(\varphi)-\alpha\left\|\varphi(0)-D\varphi(-h)\right\|^2.$$

The time derivative of the functional along the solutions of system (7.1) is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t)=-\widetilde{w}(x_t), \quad t\geq 0,$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t)+2\alpha\left[x(t)-Dx(t-h)\right]^T \\ &\quad \times\left[A_0x(t)+A_1x(t-h)+\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right].\end{aligned}$$

Observe that if $\alpha\geq 0$, then

$$\begin{aligned}&2\alpha\left[x(t)-Dx(t-h)\right]^T\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta \\ &\geq -\alpha h\left[x(t)-Dx(t-h)\right]^T\left[x(t)-Dx(t-h)\right] \\ &\quad -\alpha\int_{-h}^{0} x^T(t+\theta)G^T(\theta)G(\theta)x(t+\theta)\mathrm{d}\theta.\end{aligned}$$

This implies the inequality

$$\begin{aligned}\widetilde{w}(x_t) &\geq \left[x^T(t),x^T(t-h)\right]L(\alpha)\begin{bmatrix}x(t)\\ x(t-h)\end{bmatrix} \\ &\quad +\int_{-h}^{0} x^T(t+\theta)\left[W_2-\alpha G^T(\theta)G(\theta)\right]x(t+\theta)\mathrm{d}\theta,\end{aligned}$$

where the matrix

$$L(\alpha)=\begin{pmatrix} W_0 & 0_{n\times n}\\ 0_{n\times n} & W_1\end{pmatrix}+\alpha\begin{pmatrix} A_0+A_0^T-hI & A_1-A_0^TD-hD\\ A_1^T-D^TA_0-hD^T & -D^TA_1-A_1^TD-hD^TD\end{pmatrix}.$$

It is evident that there exists $\alpha = \alpha_1 > 0$ such that the following conditions hold.

1. The matrix $L(\alpha_1)$ is positive definite.
2. The matrix $W_2 - \alpha_1 G^T(\theta)G(\theta)$ is positive definite for $\theta \in [-h, 0]$.

For $\alpha = \alpha_1$ the inequality $\widetilde{w}(x_t) \geq 0$ holds, and we conclude that

$$\widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0.$$

This means that

$$\alpha_1 \|\varphi(0) - D\varphi(-h)\|^2 \leq v(\varphi).$$

□

Lemma 7.4. *Let system (7.1) satisfy the Lyapunov condition. Given the symmetric matrices W_0, W_1, and W_2, there exists $\alpha_2 > 0$ such that functional (7.28) satisfies the inequality*

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC^1([-h, 0], R^n).$$

Proof. Let us introduce the quantities

$$u_0 = \sup_{\tau \in (0,h)} \|U(\tau)\|, \quad u_1 = \sup_{\tau \in (0,h)} \left\|U'(\tau)\right\|, \quad u_2 = \sup_{\tau \in (0,h)} \left\|U''(\tau)\right\|$$

and

$$a_1 = \|A_1\|, \quad d = \|A_1\|, \quad g = \int_{-h}^0 \|G(\theta)\| \mathrm{d}\theta.$$

The first term in (7.29) admits the upper estimation

$$R_1 = [\varphi(0) - D\varphi(-h)]^T U(0) [\varphi(0) - D\varphi(-h)] \leq (1 + \mathrm{d})^2 u_0 \|\varphi\|_h^2.$$

The second term in (7.29),

$$R_2 = 2[\varphi(0) - D\varphi(-h)]^T \int_{-h}^0 \left[U(-h-\theta)A_1 + U'(-h-\theta)D \right.$$
$$\left. + \int_{-h}^\theta U(-\theta+\xi)G(\xi)\mathrm{d}\xi \right] \varphi(\theta)\mathrm{d}\theta,$$

can be estimated as follows:

$$R_2 \leq 2h(1+d)(a_1 u_0 + d u_1 + g u_0) \|\varphi\|_h^2.$$

The third term,

$$R_3 = -\int_{-h}^{0} \varphi^T(\theta_1) D^T \left(\int_{-h}^{\theta_1 - 0} U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right.$$
$$\left. + \int_{\theta_1 + 0}^{0} U''(\theta_1 - \theta_2) D\varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1,$$

admits the estimation

$$R_3 \leq h^2 d^2 u_2 \|\varphi\|_h^2.$$

The next term,

$$R_4 = \int_{-h}^{0} \varphi^T(\theta) \left(W_1 + (h+\theta) W_2 - D^T \left[U'(+0) - U'(-0) \right] D \right) \varphi(\theta) \mathrm{d}\theta,$$

can be estimated as

$$R_4 \leq h \left(\|W_1\| + h \|W_2\| \right) + 2d^2 u_1 \right) \|\varphi\|_h^2.$$

For the term

$$R_5 = \int_{-h}^{0} \varphi^T(\theta_1) \left(\int_{-h}^{0} \left[A_1^T U(\theta_1 - \theta_2) A_1 + A_1^T U'(\theta_1 - \theta_2) D \right. \right.$$
$$\left. \left. - D^T U'(\theta_1 - \theta_2) A_1 \right] \varphi(\theta_2) \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$

we obtain

$$R_5 \leq h^2 \left(a_1^2 u_0 + 2a_1 d u_1 \right) \|\varphi\|_h^2.$$

The term

$$R_6 = \int_{-h}^{0} \varphi^T(\theta_1) \left\{ \int_{-h}^{0} \left(\int_{-h}^{\theta_1} G^T(\xi_1) \right. \right.$$
$$\times \left. \left. \left[\int_{-h}^{\theta_2} U(\theta_1 - \xi_1 - \theta_2 + \xi_2) G(\xi_2) \mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \right) \varphi(\theta_2) \mathrm{d}\theta_2 \right\} \mathrm{d}\theta_1$$

admits the upper bound

$$R_6 \le h^2 g^2 u_0 \|\varphi\|_h^2 .$$

For the sum of the last two terms

$$\begin{aligned} R_7+R_8 &= 2\int_{-h}^{0} \varphi^T(\theta_1)\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2} A_1^T U(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\ &\quad -2\int_{-h}^{0} \varphi^T(\theta_1)\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2} D^T U'(\theta_1+h-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right)\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \end{aligned}$$

we have the following upper bound:

$$R_7+R_8 \le 2h^2 g(a_1u_0+du_1)\|\varphi\|_h^2 .$$

As a result, we arrive at the following quadratic upper bound for the functional:

$$v(\varphi) \le \alpha_2 \|\varphi\|_h^2 ,$$

with

$$\begin{aligned} \alpha_2 &= h(\|W_1\|+h\|W_2\|)+(1+d+ha_1+hg)^2 u_0 \\ &\quad +2hd\,[1+2d+ha_1+hg]\,u_1+h^2d^2u_2. \end{aligned}$$ □

We present here new upper and lower quadratic bounds for functional (7.28).

Lemma 7.5. *Let system (7.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, there exist $\beta_j>0$, $j=1,2$, such that the complete type functional (7.28) satisfies the inequality*

$$\beta_1\|\varphi(0)-D\varphi(-h)\|^2+\beta_2\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta \le v(\varphi), \quad \varphi\in PC^1([-h,0],R^n).$$

Proof. Consider the functional

$$\widetilde{v}(\varphi)=v(\varphi)-\beta_1\|\varphi(0)-D\varphi(-h)\|^2-\beta_2\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta .$$

Its time derivative along the solutions of system (7.1) is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t)=-\widetilde{w}(x_t),$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\beta_1 \left[x(t) - Dx(t-h)\right]^T \\ &\quad \times \left[A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right] \\ &\quad + \beta_2 \left[\|x(t)\|^2 - \|x(t-h)\|^2\right] \\ &\geq \left[x^T(t), x^T(t-h)\right] L(\beta_1,\beta_2) \begin{bmatrix} x(t) \\ x(t-h)\end{bmatrix} \\ &\quad + \int_{-h}^{0} x^T(t+\theta)\left[W_2 - \beta_1 G^T(\theta)G(\theta)\right] x(t+\theta)\mathrm{d}\theta.\end{aligned}$$

Here

$$\begin{aligned}L(\beta_1,\beta_2) &= \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} \\ &\quad + \beta_1 \begin{pmatrix} A_0 + A_0^T - hI & A_1 - A_0^T D - hD \\ A_1^T - D^T A_0 - hD^T & -D^T A_1 - A_1^T D - hD^T D \end{pmatrix} \\ &\quad + \beta_2 \begin{pmatrix} I & 0_{n\times n} \\ 0_{n\times n} & -I \end{pmatrix}.\end{aligned}$$

It is evident that there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that

1. The matrix $L(\beta_1,\beta_2)$ is positive definite;
2. The matrix $W_2 - \beta_1 G^T(\theta)G(\theta)$ is positive definite for $\theta \in [-h,0]$.

For these values of β_1 and β_2, $\widetilde{w}(x_t) \geq 0$ and

$$\widetilde{v}(\varphi) = \int_0^{\infty} \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0.$$

The last inequality proves the statement of the lemma. □

Lemma 7.6. *Let system (7.1) satisfy the Lyapunov condition. Given symmetric matrices W_0, W_1, and W_2, there exist $\delta_j > 0$, $j = 1,2$, such that functional (7.28) satisfies the inequality*

$$v(\varphi) \leq \delta_1 \|\varphi(0) - D\varphi(-h)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in C^1([-h,0],R^n).$$

Proof. Using the notations introduced in the proof of Lemma 7.4 we may derive the following inequalities:

$$R_1 = [\varphi(0) - D\varphi(-h)]^T U(0) [\varphi(0) - D\varphi(-h)] \le u_0 \|\varphi(0) - D\varphi(-h)\|^2,$$

$$R_2 \le [(a_1 + g)u_0 + du_1] \left(h \|\varphi(0) - D\varphi(-h)\|^2 + \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \right),$$

$$R_3 \le ha_1 (a_1 u_0 + 2du_1) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

$$R_4 + R_5 \le d^2 (2u_1 + hu_2) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

$$R_6 + R_7 + R_8 \le hg (2a_1 u_0 + gu_0 + 2du_1) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

And we arrive at the upper estimation of functional (7.29):

$$v(\varphi) \le \delta_1 \|\varphi(0) - D\varphi(-h)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

with

$$\begin{aligned} \delta_1 &= [1 + h(a_1 + g)] u_0 + hdu_1, \\ \delta_2 &= \|W_1\| + h\|W_2\| + (a_1 + g)[1 + h(a_1 + g)] u_0 \\ &\quad + d[1 + 2d + 2h(a_1 + g)] u_1 + hu_2. \end{aligned}$$

□

7.10 The $\mathcal{H}_2$ Norm of a Transfer Matrix

We compute here the value of the $\mathcal{H}_2$ norm of the transfer matrix of an exponentially stable control system of the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[x(t) - Dx(t-h)] &= A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} G(\theta) x(t+\theta) \mathrm{d}\theta \\ &\quad + B_0 u(t) + B_1 u(t-h) \end{aligned}$$

$$+\int_{-h}^{0} Q(\theta)u(t+\theta)\mathrm{d}\theta, \quad t \geq 0, \tag{7.30}$$
$$y(t) = Cx(t-h).$$

The transfer matrix of the system is of the form

$$F(s) = \mathrm{e}^{-sh}CH(s)B(s),$$

where the matrix $H(s)$ is the Laplace image of the fundamental matrix $K(t)$ of control system (7.30),

$$H(s) = \int_{0}^{\infty} K(t)\mathrm{e}^{-st}\mathrm{d}t = \left(sI - s\mathrm{e}^{-sh}D - A_0 - \mathrm{e}^{-hs}A_1 - \int_{-h}^{0} \mathrm{e}^{\theta s}G(\theta)\mathrm{d}\theta\right)^{-1},$$

and the matrix

$$B(s) = B_0 + \mathrm{e}^{-hs}B_1 + \int_{-h}^{0} \mathrm{e}^{\theta s}Q(\theta)\mathrm{d}\theta.$$

The $\mathcal{H}_2$ norm of the transfer matrix is defined as follows:

$$\begin{aligned}\|F\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathrm{Trace}\left\{F^T(\xi)F(-\xi)\right\}\mathrm{d}\xi \\ &= \mathrm{Trace}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} B^T(\xi)H^T(\xi)C^TCH(-\xi)B(-\xi)\mathrm{d}\xi\right\}.\end{aligned}$$

Applying Corollary 7.1 we arrive at the equality

$$\begin{aligned}\|F\|_{\mathcal{H}_2}^2 = {}& \mathrm{Trace}\left\{B_0^TU(0)B_0 + B_1^TU(0)B_1 + 2B_1^TU(h)B_0\right\} \\ &+2\int_{-h}^{0} \mathrm{Trace}\left\{\left[B_0^TU(\theta) + B_1^TU(h+\theta)\right]Q(\theta)\right\}\mathrm{d}\theta \\ &+\int_{-h}^{0}\int_{-h}^{0} \mathrm{Trace}\left\{Q^T(\theta_1)U(\theta_2-\theta_1)Q(\theta_2)\right\}\mathrm{d}\theta_2\mathrm{d}\theta_1,\end{aligned}$$

where $U(\tau)$ is the Lyapunov matrix of system (7.30) associated with the matrix $W = C^TC$.

References

1. Ahlfors, L.V.: Complex Analysis. McGraw-Hill, New York (1979)
2. Arnold, V.I.: Differential Equations. MIT Press, Cambridge, MA (1978)
3. Bellman, R., Cooke, K.L.: Differential-Difference Equations. Academic, New York (1963)
4. Castelan, W.B., Infante, E.F.: On a functional equation arising in the stability theory of difference-differential equations. Q. Appl. Math. **35**, 311–319 (1977)
5. Castelan, W.B., Infante, E.F.: A Liapunov functional for a matrix neutral difference-differential equation with one delay. J. Math. Anal. Appl. **71**, 105–130 (1979)
6. Curtain, R.F., Pritchard, A.J.: Infinite-dimensional linear systems theory. In: Lecture Notes in Control and Information Sciences, vol. 8. Springer, Berlin (1978)
7. Datko, R.: An algorithm for computing Liapunov functionals for some differential difference equations. In: Weiss, L. (ed.) Ordinary Differential Equations, pp. 387–398. Academic, New York (1972)
8. Datko, R.: A procedure for determination of the exponential stability of certain diferential-difference equations. Q. Appl. Math. **36**, 279–292 (1978)
9. Datko, R.: Lyapunov functionals for certain linear delay-differential equations in a Hilbert space. J. Math. Anal. Appl. **76**, 37–57 (1980)
10. Diekmann, O., von Gils, A.A., Verduyn-Lunel, S.M., Walther, H.-O.: Delay equations, functional-, complex- and nonlinear analysis. Applied Mathematics Sciences Series, vol. 110. Springer, New York (1995)
11. Driver, R.D.: Ordinary and Delay Differential Equations. Springer, New York (1977)
12. Garcia-Lozano, H., Kharitonov, V.L.: Numerical computation of time delay Lyapunov matrices. In: 6th IFAC Workshop on Time Delay Systems, L'Aquila, Italy, 10–12 July 2006
13. Golub, G.H., van Loan, C.F.: Matrix Computations. Johns Hopkins University Press, Baltimore, MD (1983)
14. Gorecki, H., Fuksa, S., Grabowski, P., Korytowski, A.: Analysis and Synthesis of Time-Delay Systems. Polish Scientific Publishers, Warsaw (1989)
15. Graham, A.: Kronecker Products and Matrix Calculus with Applications. Ellis Horwood, Chichester, UK (1981)
16. Gu, K.: Discretized Lyapunov functional for uncertain systems with multiple time-delay. Int. J. Contr. **72**, 1436–1445 (1999)
17. Gu, K., Han, Q.-L., Luo, A.C.J., Niculescu, S.-I.: Discretized Lyapunov functional for systems with distributed delay and piecewise constant coefficients. Int. J. Control **74**, 737–744 (2001)
18. Gu, K., Kharitonov, V.L., Chen, J.: Stability of Time Delay Systems. Birkhauser, Boston, MA (2003)
19. Halanay, A.: Differential Equations: Stability, Oscillations, Time Lags. Academic, New York (1966)

V.L. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*,
Control Engineering, DOI 10.1007/978-0-8176-8367-2,

20. Halanay, A., Yorke, J.A.: Some new results and problems in the theory of differential-delay equatons. SIAM Rev. **31**, 55–80 (1971)
21. Hale, J.K.: Theory of Functional Differential Equations. Springer, New York (1971)
22. Hale, J.K., Infante, E.F., Tsen, F.S.P.: Stability in linear delay equations. J. Math. Anal. Appl. **105**, 533–555 (1985)
23. Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Springer, New York (1993)
24. Hinrichsen, D., Pritchard, A.J.: Mathematical Systems Theory 1: Modelling, State Space Analysis, Stability and Robustness. Springer, Heidelberg (2005)
25. Horn, R.A., Johnson, C.A.: Matrix Analysis. Cambridge University Press, Cambridge, UK (1985)
26. Huang, W.: Generalization of Liapunov's theorem in a linear delay system. J. Math. Anal. Appl. **142**, 83–94 (1989)
27. Infante, E.F.: Some results on the Lyapunov stability of functional equations. In: Hannsgen, K.B., Herdmn, T.L., Stech, H.W., Wheeler, R.L. (eds.) Volterra and Functional Differential Equations. Lecture Notes in Pure and Applied Mathematics, vol. 81, pp. 51–60. Marcel Dekker, New York (1982)
28. Infante, E.F., Castelan, W.V.: A Lyapunov functional for a matrix difference-differential equation. J. Differ. Equat. **29**, 439–451 (1978)
29. Jarlebring, E., Vanbiervliet, J., Michiels, W.: Characterizing and computing the $\mathcal{H}_2$ norm of time delay systems by solwing the delay Lyapunov equation. In: Proceedings of the 49th IEEE Conference on Decision and Control (2010)
30. Kailath, T.: Linear Systems. Prentice-Hall, Engelewood Cliffs, NJ (1980)
31. Kharitonov, V.L.: Robust stability analysis of time delay systems: a survey. Annu. Rev. Control **23**, 185–196 (1999)
32. Kharitonov, V.L.: Lyapunov functionals and Lyapunov matrices for neutral type time-delay stystems: a single delay case. Int. J. Control **78**, 783–800 (2005)
33. Kharitonov, V.L.: Lyapunov matrices for a class of time delay systems. Syst. Control Lett. **55**, 610–617 (2006)
34. Kharitonov, V.L.: Lyapunov matrices for a class of neutral type time delay systems. Int. J. Control **81**, 883–893 (2008)
35. Kharitonov, V.L.: Lyapunov matrices: Existence and uniqueness issues. Automatica **46**, 1725–1729 (2010)
36. Kharitonov, V.L.: Lyapunov functionals and matrices. Ann. Rev. Control **34**, 13–20 (2010)
37. Kharitonov, V.L.: On the uniqueness of Lyapunov matrices for a time-delay system. Syst. Control Lett. **61**, 397–402 (2012)
38. Kharitonov, V.L., Hinrichsen, D.: Exponential estimates for time delay systems. Syst. Control Lett. **53**, 395–405 (2004)
39. Kharitonov, V.L., Mondie, S., Ochoa, G.: Frequency stability analysis of linear systems with general distributed delays. Lect. Notes Control Inf. Sci. **388**, 61–71 (2009)
40. Kharitonov, V.L., Plischke, E.: Lyapunov matrices for time delay systems. Syst. Control Lett. **55**, 697–706 (2006)
41. Kharitonov, V.L., Zhabko, A.P.: Robust stability of time-delay systems. IEEE Trans. Auto.. Control **39**, 2388–2397 (1994)
42. Kharitonov, V.L., Zhabko, A.P.: Lyapunov-Krasovskii approach to robust stability analysis of time delay systems. Automatica **39**, 15–20 (2003)
43. Kolmanovskii, V., Myshkis, A.: Applied Theory of Functional Differential Equations. Kluwer, Dordrecht, the Netherlands (1992)
44. Kolmanovskii, V.B., Nosov, V.R.: Stability of Functional Differential Equations. Mathematics in Science and Engineering, vol. 180. Academic, New York (1986)
45. Kolmogorov, A., Fomin, S.: Elements of the Theory of Functions and Functional Analysis. Greylock, Rochester, NY (1961)
46. Krasovskii, N.N.: Stability of Motion. [Russian], Moscow, 1959 [English translation]. Stanford University Press, Stanford, CA (1963)

47. Krasovskii, N.N.: On using the Lyapunov second method for equations with time delay [Russian]. Prikladnaya Matematika i Mekhanika. **20**, 315–327 (1956)
48. Krasovskii, N.N.: On the asymptotic stability of systems with aftereffect [Russian]. Prikladnaya Matematika i Mekhanika. **20**, 513–518 (1956)
49. Lakshmikantam, V., Leela, S.: Differential and Integral Inequalities. Academic, New York (1969)
50. Levinson, N., Redheffer, R.M.: Complex Variables. Holden-Day, Baltimore, MD (1970)
51. Louisel, J.: Growth estimates and asymptotic stability for a class of differential-delay equation having time-varying delay. J. Math. Anal. Appl. **164**, 453–479 (1992)
52. Louisell, J.: Numerics of the stability exponent and eigenvalue abscissas of a matrix delay system. In: Dugard, L., Verriest, E.I. (eds.) Stability and Control of Time-delay Systems. Lecture Notes in Control and Information Sciences, vol. 228, pp. 140–157. Springer, New York (1997)
53. Louisell, J.: A matrix method for determining the imaginary axis eigenvalues of a delay system. IEEE Trans. Autom. Control **46**, 2008–2012 (2001)
54. Malek-Zavarei, M., Jamshidi, M.: Time delay systems: analysis, optimization and applications. North-Holland Systems and Control Series, vol. 9. North-Holland, Amsterdam (1987)
55. Marshall, J.E., Gorecki, H., Korytowski, A., Walton, K.: Time-Delay Systems: Stability and Performance Criteria with Applications. Ellis Horwood, New York (1992)
56. Mondie, S.: Assesing the exact stability region of the single delay scalar equation via its Lyapunov function. IMA J. Math. Control Inf. (2012). doi: ID:DNS004
57. Myshkis, A.D.: General theory of differential equations with delay [Russian]. Uspekhi Matematicheskikh Nauk. **4**, 99–141 (1949)
58. Niculescu, S.-I.: Delay Effects on Stability: A Robust Control Approach. Springer, Heidelberg (2001)
59. Ochoa, G., Mondie, S., Kharitonov, V.L.: Time delay systems with distributed delays: critical values. In: Proceedings of the 8th IFAC Workshop on Time Delay Systems, Sinaia, Romania, 1–3 Sept 2009
60. Plishke, E.: Transient effects of linear dynamical systems. Ph.D. thesis, University of Bremen, Bremen, Germany (2005)
61. Razumikhin, B.S.: On the stability of systems with a delay [Russian]. Prikladnaya Matematika i Mekhanika. **20**, 500–512 (1956)
62. Razumikhin, B.S.: Application of Liapunov's method to problems in the stability of systems with a delay [Russian]. Automatika i Telemekhanika. **21**, 740–749 (1960)
63. Repin, Yu.M.: Quadratic Lyapunov functionals for systems with delay [Russian]. Prikladnaya Matematika i Mekhanika. **29**, 564–566 (1965)
64. Richard, J.-P.: Time-delay systems: an overview of some recent advances and open problems. Automatica **39**, 1667–1694 (2003)
65. Rudin W.: Real and Complex Analysis. McGraw-Hill, New York (1973)
66. Rudin, W.: Functional Analysis. McGraw-Hill, New York (1987)
67. Stépán, G.: Retarded Dynamical Systems: Stability and Characteristic Function. Wiley, New York (1989)
68. Velazquez-Velazquez, J., Kharitonov, V.L.: Lyapunov-Krasovskii functionals for scalar neutral type time delay equation. Syst. Control Lett. **58**, 17–25 (2009)
69. Volterra, V.: Sulle equazioni integrodifferenciali della teorie dell'elasticita. Atti. Accad. Lincei. **18**, 295 (1909)
70. Volterra, V.: Theorie mathematique de la lutte pour la vie [French]. Gauthier-Villars, Paris (1931)
71. Zhou, K., Doyle, J.C., Glover, K.: Robust and Optimal Control. Prentice-Hall, Upper Saddle River, NJ (1996)
72. Zubov, V.I.: The Methods of A.M. Lyapunov and Their Applications. Noordhoff, Groningen, the Netherlands (1964)

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